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All Exercises

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Contents

1 Basics of Quantum Optics

1.1 Coherent state

Calculate the expectation value of the photon number operator $\hat{n}$ for a coherent state

$$\langle\alpha|\hat{n}|\alpha\rangle \tag{1.1}$$

and its fluctuation

$$\Delta N = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2}. \tag{1.2}$$

Prove that for a measurement of the photon number in the field given by state, the probability of detecting n photons follows a Poissonian distribution

$$P_n = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!}, \tag{1.3}$$

where λ is the mean value of the Poissonian distribution.

Solution

The (normalized) coherent state was derived in the seminar as [1]

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.4}$$

We can now calculate the expectation value of the photon number operator $\hat{n}$ as

$$\begin{aligned}
\langle\alpha|\hat{n}|\alpha\rangle &= \left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\right)\hat{n}\left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle\right) \\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\langle n|\,\hat{n}\,|m\rangle\right) \\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}m\underbrace{\langle n|m\rangle}_{\delta_{mn}}\right) \\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}n\right) = \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^n}{(n-1)!}\right) \\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2(n+1)}}{n!}\right) = |\alpha|^2\mathrm{e}^{-|\alpha|^2}\underbrace{\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}\right)}_{\exp(|\alpha|^2)} \\
&= |\alpha|^2.
\end{aligned} \tag{1.5}$$

[1] It can also be found in the *Quantum optics* script in chapter 6.1.4.

In the second to last line we used an index shift $n \to n+1$. Now in order to calculate the fluctuation of the photon number we need to calculate $\langle\alpha|\hat{n}^2|\alpha\rangle$ as well

$$
\begin{aligned}
\langle\alpha|\hat{n}^2|\alpha\rangle &= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\langle n|\,\hat{n}^2\,|m\rangle\right)\\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}m^2\underbrace{\langle n|m\rangle}_{\delta_{mn}}\right)\\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}n^2\right) = \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{n}}{(n-1)!}n\right)\\
&= \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2(n+1)}}{n!}(n+1)\right) = \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2(n+1)}}{n!}\right)+\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2(n+1)}}{(n-1)!}\right)\\
&= \underbrace{\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2(n+1)}}{n!}\right)}_{|\alpha|^2}+\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\frac{|\alpha|^{2(n+2)}}{(n)!}\right)\\
&= |\alpha|^2+|\alpha|^4.
\end{aligned}
\tag{1.6}
$$

Now we can simply calculate the fluctuations using (1.5)^2 and (1.6)

$$
\Delta N = \sqrt{\langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2} = \sqrt{|\alpha|^2+|\alpha|^4-|\alpha|^4} = |\alpha|. \tag{1.7}
$$

The probability of detecting n photons can be calculated as follows:

$$
P_n = |\langle n|\alpha\rangle|^2. \tag{1.8}
$$

Here, $\langle n|\alpha\rangle$ describes the projection of the coherent state $|\alpha\rangle$ onto the number state with n photons. The probability is then given as the magnitude squared. First lets calculate

$$
\langle n|\alpha\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\underbrace{\langle n|m\rangle)}_{\delta_{mn}} = \mathrm{e}^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{\sqrt{n!}}. \tag{1.9}
$$

The the probability P_n is

$$
P_n = |\mathrm{e}^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{\sqrt{n!}}|^2 = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = \mathrm{e}^{-\lambda}\frac{\lambda^n}{n!} \tag{1.10}
$$

where $\lambda = |\alpha|^2$ according to (1.5).

Remark: It is much easier to use the defining property of the coherent state, namely that $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ and correspondingly $\langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*$. Then we have e. g.

$$
\begin{aligned}
\langle\alpha|\hat{n}|\alpha\rangle &= \left\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\right\rangle = \alpha^*\alpha\langle\alpha|\alpha\rangle = |\alpha|^2\\
\langle\alpha|\hat{n}^2|\alpha\rangle &= \left\langle\alpha|\hat{a}^\dagger\underbrace{\hat{a}\hat{a}^\dagger}_{=1+\hat{a}^\dagger\hat{a}}\hat{a}|\alpha\right\rangle = \left\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\right\rangle + \left\langle\alpha|\hat{a}^{\dagger 2}\hat{a}^2|\alpha\right\rangle = |\alpha|^2+|\alpha|^4.
\end{aligned}
\tag{1.11}
$$

1.2 Fluctuations of the electric field

Prove that the fluctuations of the electric field for a coherent state are identical to the fluctuations of a vacuum state, meaning:

$$\Delta E_x = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} =: \xi. \tag{1.12}$$

Consider an electric field operator given by:

$$\hat{E}(x,t) = \mathrm{i}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger \mathrm{e}^{\mathrm{i}\omega t}\right). \tag{1.13}$$

Solution

The fluctuations can be analaogously determined by using (1.2). First we start with

$$\begin{aligned}\langle\alpha|\hat{E}|\alpha\rangle &= \left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\right)\hat{E}\left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle\right)\\ &= \mathrm{i}\xi\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\langle n|\,\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger\mathrm{e}^{\mathrm{i}\omega t}\,|m\rangle\right). \end{aligned}\tag{1.14}$$

Now we can use the properties of the creation operator $\hat{a}^\dagger$ and annihilation operator $\hat{a}$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle \tag{1.15}$$

to calculate the term

$$\langle n|\,\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger\mathrm{e}^{\mathrm{i}\omega t}\,|m\rangle = \mathrm{e}^{-\mathrm{i}\omega t}\sqrt{m}\underbrace{\langle n|m-1\rangle}_{\delta_{n,m-1}} - \mathrm{e}^{\mathrm{i}\omega t}\sqrt{m+1}\underbrace{\langle n|m+1\rangle}_{\delta_{n,m+1}}. \tag{1.16}$$

Now we can substitute (1.16) back into (1.14) and use the relation

$$\sum_{n=0}^{\infty} f(n)\delta_{m,n} = f(m), \quad \text{where} \quad f(n) \quad \text{is a generic function} \tag{1.17}$$

to eliminate the summation over n:

$$\begin{aligned}\langle\alpha|\hat{E}|\alpha\rangle &= \mathrm{i}\xi\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^m}{\sqrt{m!}}\langle n|\,\mathrm{e}^{-\mathrm{i}\omega t}\delta_{n,m-1} - \mathrm{e}^{\mathrm{i}\omega t}\delta_{n,m+1}\,|m\rangle\right)\\ &= \mathrm{i}\xi\mathrm{e}^{-|\alpha|^2}\sum_{m=0}^{\infty}\left(\frac{\alpha^{*m-1}}{\sqrt{(m-1)!}}\frac{\alpha^m}{\sqrt{m!}}\sqrt{m}\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\alpha^{*m+1}}{\sqrt{(m+1)!}}\frac{\alpha^m}{\sqrt{m!}}\sqrt{m+1}\mathrm{e}^{\mathrm{i}\omega t}\right)\\ &= \mathrm{i}\xi\mathrm{e}^{-|\alpha|^2}\sum_{m=0}^{\infty}\left(\frac{\alpha^{*m-1}}{\sqrt{(m-1)!}}\frac{\alpha^m}{\sqrt{(m-1)!}}\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\alpha^{*m+1}}{\sqrt{m!}}\frac{\alpha^m}{\sqrt{m!}}\mathrm{e}^{\mathrm{i}\omega t}\right)\\ &= \mathrm{i}\xi\mathrm{e}^{-|\alpha|^2}\sum_{m=0}^{\infty}\left(\frac{\alpha^{*m-1}\alpha^m}{(m-1)!}\mathrm{e}^{-\mathrm{i}\omega t} - \frac{\alpha^{*m+1}\alpha^m}{m!}\mathrm{e}^{\mathrm{i}\omega t}\right). \end{aligned}\tag{1.18}$$

Now we can again use an index shift in the first part of the sum $m \to m+1$

$$\begin{aligned}\langle\alpha|\hat{E}|\alpha\rangle &= \mathrm{i}\xi \cancel{\mathrm{e}^{-|\alpha|^2}}\Big(\alpha \underbrace{\sum_{m=0}^{\infty}\frac{|\alpha|^2}{m!}}_{\cancel{\exp(|\alpha|^2)}}\mathrm{e}^{-\mathrm{i}\omega t} - \alpha^* \underbrace{\sum_{m=0}^{\infty}\frac{|\alpha|^2}{m!}}_{\cancel{\exp(|\alpha|^2)}}\mathrm{e}^{+\mathrm{i}\omega t}\Big) \\ &= i\xi\left(\alpha \mathrm{e}^{-\mathrm{i}\omega t} - \alpha^*\mathrm{e}^{\mathrm{i}\omega t}\right). \end{aligned} \tag{1.19}$$

Now we also calculate $\langle\alpha|\hat{E}^2|\alpha\rangle$

$$\begin{aligned}\langle\alpha|\hat{E}^2|\alpha\rangle &= \left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\langle n|\right)\hat{E}^2\left(\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\alpha^{m}}{\sqrt{m!}}|m\rangle\right) \\ &= -\xi^2\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{m}}{\sqrt{m!}}\langle n|\,(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger\mathrm{e}^{\mathrm{i}\omega t})^2\,|m\rangle\right). \end{aligned} \tag{1.20}$$

Now we explicitly try to simplify the operator in the bracket

$$\begin{aligned}(\hat{a}\mathrm{e}^{-\mathrm{i}\omega t} - \hat{a}^\dagger\mathrm{e}^{\mathrm{i}\omega t})^2 &= \hat{a}^2\mathrm{e}^{-2\mathrm{i}\omega t} + \hat{a}^{\dagger 2}\mathrm{e}^{2\mathrm{i}\omega t} - \underbrace{(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})}_{=2\hat{a}^\dagger\hat{a}+[\hat{a},\hat{a}^\dagger]=2\hat{n}+1} \\ &= \hat{a}^2\mathrm{e}^{-2\mathrm{i}\omega t} + \hat{a}^{\dagger 2}\mathrm{e}^{2\mathrm{i}\omega t} - 2\hat{n} - 1. \end{aligned} \tag{1.21}$$

Since the state $|\alpha\rangle$ is normalized, we have $\langle\alpha|1|\alpha\rangle = 1$. The expectation value of $\hat{n}$ was calculated in the first task and is $\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$. Using this we can rewrite (1.20) as

$$\begin{aligned}\langle\alpha|\hat{E}^2|\alpha\rangle &= -\xi^2\mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{m}}{\sqrt{m!}}\langle n|\,(\hat{a}^2\mathrm{e}^{-2\mathrm{i}\omega t} + \hat{a}^{\dagger 2}\mathrm{e}^{2\mathrm{i}\omega t} - 2\hat{n} - 1)\,|m\rangle\right) \\ &= -\xi^2\left(-1 - 2|\alpha|^2 + \mathrm{e}^{-|\alpha|^2}\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{m}}{\sqrt{m!}}\left[\langle n|\,\hat{a}^2\mathrm{e}^{-2\mathrm{i}\omega t} + \hat{a}^{\dagger 2}\mathrm{e}^{2\mathrm{i}\omega t}\,|m\rangle\right]\right)\right). \end{aligned} \tag{1.22}$$

We now focus on the bracket term first using the properties of the ladder operators

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{m}}{\sqrt{m!}}\left[\langle n|\,\hat{a}^2\mathrm{e}^{-2\mathrm{i}\omega t} + \hat{a}^{\dagger 2}\mathrm{e}^{2\mathrm{i}\omega t}\,|m\rangle\right] \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{m}}{\sqrt{m!}}\left[\mathrm{e}^{-2\mathrm{i}\omega t}\sqrt{m}\sqrt{m-1}\,\langle n|m-2\rangle + \mathrm{e}^{2\mathrm{i}\omega t}\,\langle n|m+2\rangle\right] \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^{*m-2}}{\sqrt{(m-2)!}}\frac{\alpha^{m}}{\sqrt{m!}}\sqrt{m}\sqrt{m-1}\mathrm{e}^{-2\mathrm{i}\omega t} + \frac{\alpha^{*m+2}}{\sqrt{(m+2)!}}\frac{\alpha^{m}}{\sqrt{m!}}\sqrt{m+1}\sqrt{m+2}\mathrm{e}^{2\mathrm{i}\omega t}\right) \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^{*m-2}}{\sqrt{(m-2)!}}\frac{\alpha^{m}}{\sqrt{(m-2)!}}\mathrm{e}^{-2\mathrm{i}\omega t} + \frac{\alpha^{*m+2}}{\sqrt{m!}}\frac{\alpha^{m}}{\sqrt{m!}}\mathrm{e}^{2\mathrm{i}\omega t}\right) \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^{*m-2}\alpha^{m}}{(m-2)!}\mathrm{e}^{-2\mathrm{i}\omega t} + \frac{\alpha^{*m+2}\alpha^{m}}{m!}\mathrm{e}^{2\mathrm{i}\omega t}\right) \\ &= \sum_{m=0}^{\infty}\left(\frac{\alpha^{*m}\alpha^{m}}{m!}\alpha^2\mathrm{e}^{-2\mathrm{i}\omega t} + \frac{\alpha^{*m}\alpha^{m}}{m!}\alpha^{*2}\mathrm{e}^{2\mathrm{i}\omega t}\right) = \exp\left(|\alpha|^2\right)(\alpha^2\mathrm{e}^{-2\mathrm{i}\omega t} + \alpha^{*2}\mathrm{e}^{2\mathrm{i}\omega t}). \end{aligned} \tag{1.23}$$

Substituting (1.23) into (1.22) yields

$$\langle\alpha|\hat{E}^2|\alpha\rangle = -\xi^2\left(-1 - 2|\alpha|^2 + \alpha^2\mathrm{e}^{-2\mathrm{i}\omega t} + \alpha^{*2}\mathrm{e}^{2\mathrm{i}\omega t}\right). \tag{1.24}$$

Furthermore we can also calculate $\langle\alpha|\hat{E}|\alpha\rangle^2$ by squaring (1.19)

$$\langle\alpha|\hat{E}|\alpha\rangle^2 = -\xi^2(\alpha \mathrm{e}^{-\mathrm{i}\omega t} - \alpha^* \mathrm{e}^{\mathrm{i}\omega t})^2 = -\xi^2(-2|\alpha|^2 + \alpha^2 \mathrm{e}^{-2\mathrm{i}\omega t} + \alpha^{*2} \mathrm{e}^{2\mathrm{i}\omega t}). \tag{1.25}$$

Then the fluctuations are then given by

$$\Delta\hat{E} = \sqrt{\langle\hat{E}^2\rangle - \langle\hat{E}\rangle^2} = \sqrt{\xi^2} = \xi = \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}. \tag{1.26}$$

1.3 Photon number fluctuations, phase distribution

Consider the following pure superposition state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + \mathrm{e}^{\mathrm{i}\theta}|1\rangle\Big). \tag{1.27}$$

Calculate the fluctuations of the photon number and the phase distribution associated to this state. Interpret your results.
Hint: The phase distributions associated to a particular state is given by:

$$P(\varphi) = \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2, \quad \textit{where} \quad |\varphi\rangle = \sum_n \mathrm{e}^{\mathrm{i}n\varphi}|n\rangle \tag{1.28}$$

is known as the phase state.

Solution:

For the fluctuations of the photon number we again use (1.2). We start with

$$\begin{aligned}\langle\psi|\hat{N}^2|\psi\rangle &= \frac{1}{2}\Big(\langle 1|\mathrm{e}^{-\mathrm{i}\theta} + \langle 0|\Big)\hat{N}^2\Big(|0\rangle + \mathrm{e}^{\mathrm{i}\theta}|1\rangle\Big)\\ &= \frac{1}{2}\left(\mathrm{e}^{-\mathrm{i}\theta}\underbrace{\langle 1|\hat{N}^2|0\rangle}_{=0} + \mathrm{e}^{\mathrm{i}\theta}\underbrace{\langle 0|\hat{N}^2|1\rangle}_{=0} + \underbrace{\langle 0|\hat{N}^2|0\rangle}_{=0} + \underbrace{\langle 1|\hat{N}^2|1\rangle}_{=1}\right) = \frac{1}{2}.\end{aligned} \tag{1.29}$$

The expecation value for $\hat{N}$ and $\hat{N}^2$ is indeed identical in this case. Then we find

$$\Delta\hat{N} = \sqrt{\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2} = \sqrt{\frac{1}{2} - \frac{1}{4}} = \frac{1}{2}. \tag{1.30}$$

The phase distribution can also be calculated straightforwardly

$$\begin{aligned}P(\varphi) &= \frac{1}{2\pi}\left|\sum_n \mathrm{e}^{-in\varphi}\langle n|\psi\rangle\right|^2\\ &= \frac{1}{2\pi}\left|\mathrm{e}^{-i0\varphi}\langle 0|\psi\rangle + \mathrm{e}^{-\mathrm{i}\varphi}\langle 1|\psi\rangle\right|^2 = \frac{1}{2\pi}\left|\frac{1}{\sqrt{2}} + \mathrm{e}^{-\mathrm{i}\varphi}\frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\theta}\right|^2\\ &= \frac{1}{4\pi}(1 + \mathrm{e}^{i(\theta-\varphi)})(1 + \mathrm{e}^{-i(\theta-\varphi)}) = \frac{1}{8\pi}\Big(1 + \mathrm{e}^{i(\theta-\varphi)} + \mathrm{e}^{-i(\theta-\varphi)} + 1\Big)\\ &= \frac{1}{4\pi}\big(2 + 2\cos(\theta-\varphi)\big) = \frac{1+\cos(\theta-\varphi)}{2\pi}.\end{aligned} \tag{1.31}$$

2 Basics of Nonlinear Optics

2.1 Collinear phase matching in a BBO crystal

We consider a continuous laser beam ($\lambda_p = 405\,\text{nm}$) which enters a BBO crystal. The refractive indices of this crystal are given by the Sellmeier equation

$$n^2(\lambda) = A + \frac{B}{\lambda^2 + C} + D\lambda^2. \tag{2.1}$$

Table 1: Sellmeier parameters for BBO ($T = 20\,°\text{C}$) in the range of $\lambda \approx 212 \ldots 1064\,\text{nm}$ related to equation (2.1). [Ref.: Eimerl et al., J. Appl. Phys. 62, 1968 (1987)]

	A	B [μm^2]	C [$\frac{1}{\mu\text{m}^2}$]	D [$\frac{1}{\mu\text{m}^2}$]
n_e	2.3730	0.0128	-0.0156	-0.0044
n_o	2.7405	0.0184	-0.0179	-0.0155

BBO is a negative, uniaxial crystal. For such a crystal the angle-dependent refractive index $n_e(\theta,\lambda)$ for a given propagation direction of light can be calculated with the index ellipsoid equation

$$\frac{1}{n_e^2(\theta,\lambda)} = \frac{\sin^2(\theta)}{n_e^2(\lambda)} + \frac{\cos^2(\theta)}{n_o^2(\lambda)} \tag{2.2}$$

where θ is the angle between the optical axis and the propagation direction.

Consider collinear SPDC type I ($e \rightarrow o + o$) and calculate the angles θ which fulfill the phase matching condition if

a) $\lambda_i = 2\lambda_p$

b) $\lambda_i = 3\lambda_p$

c) $\lambda_i = 4\lambda_p$.

Solution

Since we want to achieve type I phase matching we want to cut the crystal in such a way, that the refractive index of the pump wave (extraordinary beam) matches the idler wave (ordinary beam). The condition for that can be written as

$$n_o(\lambda_i) = n_e(\lambda_p,\theta). \tag{2.3}$$

Now we can use equation (2.2) to find the correct cutting angle θ

$$\begin{aligned}
\frac{1}{n_o^2(\lambda_i)} &= \frac{\sin^2(\theta)}{n_e^2(\lambda_p)} + \frac{\cos^2(\theta)}{n_o^2(\lambda_p)} \\
&= \frac{n_o^2(\lambda_p)\sin^2(\theta) + n_e^2(\lambda_p)\cos^2(\lambda_p)}{n_e^2(\lambda_p)n_o^2(\lambda_p)} \\
&= \frac{n_e^2(\lambda_p) + (n_o^2(\lambda_p)n_e^2(\lambda_p))\sin^2(\theta)}{n_e^2(\lambda_p)n_o^2(\lambda_p)} \\
\Rightarrow \sin^2(\theta) &= \left(\frac{1}{n_o^2(\lambda_i)} - \frac{1}{n_o^2(\lambda_p)}\right)\frac{n_e^2(\lambda_p)n_o^2(\lambda_p)}{n_o^2(\lambda_p) - n_e^2(\lambda_p)} \\
\Rightarrow \theta &= \arcsin\sqrt{\left(\frac{1}{n_o^2(\lambda_i)} - \frac{1}{n_o^2(\lambda_p)}\right)\frac{n_e^2(\lambda_p)n_o^2(\lambda_p)}{n_o^2(\lambda_p) - n_e^2(\lambda_p)}}. \qquad (2.4)
\end{aligned}$$

Now we can calculate the refractive indices using the Sellmeier equations: $n_o(\lambda_p) = 1.692, n_e(\lambda_p) = 1.568$ and for the three tasks $n_o(2\lambda_p) = 1.661, n_o(3\lambda_p) = 1.652$ and $n_o(4\lambda_p) = 1.645$. If we insert these values into (2.4) we obtain

a) $\theta = 28{,}67°$

b) $\theta = 33{,}02°$

c) $\theta = 36{,}38°$.

2.2 Collinear phase matching in a BBO crystal

As discussed in the lecture, phase matching can be achieved by different means in birefringent media, for example: by angle tuning, temperature tuning and by periodically poling of the non-linear crystal, also known as quasi phase matching. The last two methods will be discussed in the seminar.

The task is to research about the different types of phase matching and how their conditions are satisfied for the following scenarios:

- *In birefringent crystals through temperature tuning.*
- *In periodically poled crystals.*

Temperature tuning is used when angle tuning is not possible, e.g. when the spatial walk off due to different propagation directions of the Poynting vector and k-vector is too large. The walkoff limits the spatial overlap of the waves and decreases the efficiency of the nonlinear process. For certain crystals like lithium niobate the birefringence is strongly temperature dependent. Here it is possible to hold θ fixed at 90° and vary the temperature of the crystal to achieve phase matching.

The last type of phase matching is called quasi-phase-matching. Here we create a periodically poled material, fabricated in such a manner that the orientation of the crystal axis (of a ferromagnetic material) is inverted periodically within the material. We observe, that the nonlinear coefficient d_{eff} changes its sign periodically which can compeansate for a nonzero mismatch Δk. The periodicity of this structure is determined by the coherence length of the nonlinear medium. Ideally the periodicity is shorte than the coherence length. Then, each time the field amplitude of the generated wave is about to begin to decrease, a reversal of sign of d_{eff} occurs which allows the field amplitude to grow monotonically.

3 Correlation state

3.1 Coherence of Schrödinger cat state

Determine the coherence functions $g^{(1)}$ and $g^{(2)}$ for the superposition of two coherent states (also known as Schrödinger cat state)

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |-\alpha\rangle). \tag{3.1}$$

Solution

The first order coherence function $g^{(1)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)$ is defined as

$$\begin{aligned} g^{(1)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) &= \frac{G^{(1)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)}{\sqrt{G^{(1)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_1, t_1)G^{(1)}(\boldsymbol{r}_2, t_2, \boldsymbol{r}_2, t_2)}} \\ &= \frac{\left\langle \hat{E}^{(-)}(\boldsymbol{r}_1, t_1)\hat{E}^{(+)}(\boldsymbol{r}_2, t_2)\right\rangle}{\sqrt{\left\langle \hat{E}^{(-)}(\boldsymbol{r}_1, t_1)\hat{E}^{(+)}(\boldsymbol{r}_1, t_1)\right\rangle \left\langle \hat{E}^{(-)}(\boldsymbol{r}_2, t_2)\hat{E}^{(+)}(\boldsymbol{r}_2, t_2)\right\rangle}}. \end{aligned} \tag{3.2}$$

The coherent state $|\pm\alpha\rangle$ is defined as

$$|\pm\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n \frac{(\pm\alpha)^n}{\sqrt{n!}}. \tag{3.3}$$

The quantized electric field ist given as

$$\hat{E}(\boldsymbol{r}, t) = \mathrm{i}\sum_{k,s}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\hat{\mathbf{e}}_{k,s}\hat{a}_k \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \tag{3.4}$$

where k denotes the different frequency modes of the electric field and s the two polarization directions. The action of electric field operator on the cat state is then

$$\begin{aligned} \hat{E}(x_i, t_i)\,|\Psi\rangle &= \frac{\mathrm{i}}{\sqrt{2}}\xi \mathrm{e}^{\mathrm{i}(kx_i-\omega t_i)}\hat{a}(|\alpha\rangle + |-\alpha\rangle) \\ &= \frac{\mathrm{i}}{\sqrt{2}}\xi \mathrm{e}^{\mathrm{i}(kx_i-\omega t_i)}\alpha(|\alpha\rangle - |-\alpha\rangle). \end{aligned} \tag{3.5}$$

We can show that $\hat{a}\,|-\alpha\rangle = -\alpha\,|-\alpha\rangle$ by using another definition of the coherent state

$$|\alpha\rangle = \mathscr{D}(\alpha)\,|0\rangle \quad \text{with} \quad \mathscr{D}(\alpha) = \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha\hat{a}}. \tag{3.6}$$

We note the following property of the coherence operator $\mathscr{D}(\alpha)$:

$$\mathscr{D}(\alpha)\hat{a}\mathscr{D}^\dagger(\alpha) = \hat{a} - \alpha. \tag{3.7}$$

Then we find

$$\begin{aligned} \underbrace{\mathscr{D}^\dagger(\alpha)\mathscr{D}(\alpha)}_{\mathbb{1}}\,\hat{a}\underbrace{\mathscr{D}^\dagger(\alpha)\,|0\rangle}_{|-\alpha\rangle} &= \mathscr{D}^\dagger(\alpha)(\hat{a}-\alpha)\,|0\rangle \\ \hat{a}\,|-\alpha\rangle &= \mathscr{D}^\dagger(\alpha)(\cancel{\hat{a}|0\rangle} - \alpha\,|0\rangle) = -\alpha\,|-\alpha\rangle\,. \end{aligned} \tag{3.8}$$

Now we are ready to calculate $G^{(1)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)$

$$\begin{aligned} G^{(1)}(x_1, t_1, x_2, t_2) &= \frac{|\alpha|^2}{2}\xi^2 \mathrm{e}^{\mathrm{i}[k(x_2-x_1)-\omega(t_2-t_1)]}[(\langle\alpha| - \langle-\alpha|)(|\alpha\rangle - |-\alpha\rangle)] \\ &= \frac{|\alpha|^2}{2}\xi^2 \mathrm{e}^{\mathrm{i}[k(x_2-x_1)-\omega(t_2-t_1)]}(\langle\alpha|\alpha\rangle - \langle\alpha|-\alpha\rangle - \langle-\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle). \end{aligned} \tag{3.9}$$

Since $|\alpha\rangle$ is normalized we have $\langle\alpha|\alpha\rangle = 1$ but we still need to compute

$$\begin{aligned} \langle\alpha|-\alpha\rangle &= \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_n \frac{(\alpha^*)^n}{\sqrt{n!}}\langle n|\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_m \frac{(-\alpha)^m}{\sqrt{m!}}|m\rangle \\ &= \mathrm{e}^{-|\alpha|^2}\sum_{n,m}\frac{(-\alpha^*\alpha)^m}{\sqrt{m!n!}}\langle n|m\rangle = \mathrm{e}^{-|\alpha|^2}\sum_n \frac{(-|\alpha|^2)^n}{n!} = \mathrm{e}^{-|\alpha|^2}\mathrm{e}^{-|\alpha|^2} = \mathrm{e}^{-2|\alpha|^2}. \end{aligned} \tag{3.10}$$

$$G^{(1)}(x_1, t_1, x_2, t_2) = \frac{|\alpha|^2}{2}\xi^2 \mathrm{e}^{\mathrm{i}[k(x_2-x_1)-\omega(t_2-t_1)]}(2 - 2\mathrm{e}^{-2|\alpha|^2}) \tag{3.11}$$

$$G^{(1)}(x_i, t_i, x_i, t_i) = \frac{|\alpha|^2}{2}\xi^2(2 - 2\mathrm{e}^{-2|\alpha|^2}). \tag{3.12}$$

Finally we can compute the first order coherence function as

$$g^{(1)}(x_1, t_1, x_2, t_2) = \mathrm{e}^{\mathrm{i}[k(x_2-x_1)-\omega(t_2-t_1)]} \quad \Rightarrow \quad |g^{(1)}|^2 = 1. \tag{3.13}$$

Analogously we can compute the second order coherence function as

$$\begin{aligned} g^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2) &= \frac{G^{(2)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_2, t_2)}{\sqrt{G^{(1)}(\boldsymbol{r}_1, t_1, \boldsymbol{r}_1, t_1)G^{(1)}(\boldsymbol{r}_2, t_2, \boldsymbol{r}_2, t_2)}} \\ &= \frac{\left\langle \hat{E}^{(-)}(\boldsymbol{r}_1, t_1)\hat{E}^{(-)}(\boldsymbol{r}_2, t_2)\hat{E}^{(+)}(\boldsymbol{r}_1, t_1)\hat{E}^{(+)}(\boldsymbol{r}_2, t_2)\right\rangle}{\sqrt{\left\langle \hat{E}^{(-)}(\boldsymbol{r}_1, t_1)\hat{E}^{(+)}(\boldsymbol{r}_1, t_1)\right\rangle\left\langle \hat{E}^{(-)}(\boldsymbol{r}_2, t_2)\hat{E}^{(+)}(\boldsymbol{r}_2, t_2)\right\rangle}}. \end{aligned} \tag{3.14}$$

After some calculation we find, that the absolute value of the second order coherence function is also equal to one, which means that this state is fully coherent. Indeed, all higher order coherence functions are also one.

3.2 Second order correlation function

A light source emits a train of single photons with exactly regular time intervals between them. Sketch the $g^{(2)}$ function that would be expected for the following cases:

1. *When the time interval between the photons is much larger than the response time τ_D of the detector*
2. *When the time interval is much smaller than τ_D.*

4 Interference experiments

4.1 Mach-Zehnder Interferometer

Consider the Mach-Zehnder interferometer of Fig. 1, with the following input state at the first beam splitter

$$|in\rangle = \frac{1}{\sqrt{2}}|0\rangle_0 (|\alpha\rangle_1 + |-\alpha\rangle_1), \tag{4.1}$$

where $|\alpha|$ is assumed to be large enough so that $\langle -\alpha|\alpha\rangle \approx 0$.
Obtain the expectation value of the number difference operator

$$\hat{O} = \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}, \tag{4.2}$$

where a is the spatial mode related to the output port of BS2 towards detector D1 and b towards D2. Examine the uncertainty in the measurement of the phase shift θ for this case of input state. Compare the result obtained using the coherent state in one of the input modes.

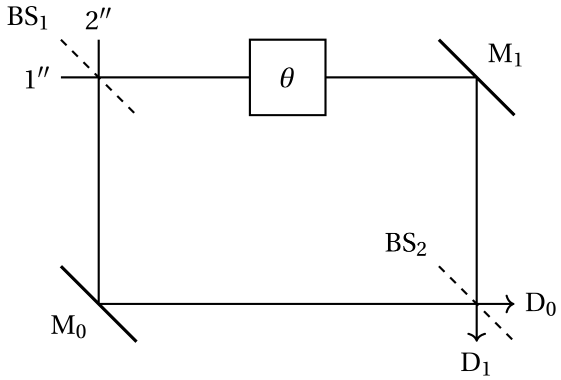

Fig. 1: A Mach-Zehnder interferometer with an input state given by $|\text{in}\rangle$, the beam splitters BS1 and BS2, M1 and M2 are mirrors and the box labeled θ represents the relative phase shift between the two arms.

Solution

First we want to calculate the state at the output of the second beam splitter. First we express the coherent states using the coherence operator (3.6)

$$|\text{in}\rangle = \frac{1}{\sqrt{2}}\Big(\exp\Big(\alpha a_{1''}^\dagger - \alpha^* a_{1''}\Big) + \exp\Big(-\alpha a_{1''}^\dagger + \alpha^* a_{1''}\Big)\Big)|0\rangle_{1''}|0\rangle_{2''}$$

$$|\text{BS1}\rangle = \frac{1}{\sqrt{2}}\Bigg(\exp\Bigg(\frac{\alpha}{\sqrt{2}}(a_{1'}^\dagger + \mathrm{i}a_{2'}^\dagger) - \frac{\alpha^*}{\sqrt{2}}(a_{1'} - \mathrm{i}a_{2'})\Bigg)$$
$$+ \exp\Bigg(\frac{-\alpha}{\sqrt{2}}(a_{1'}^\dagger + \mathrm{i}a_{2'}^\dagger) - \frac{-\alpha^*}{\sqrt{2}}(a_{1'} - \mathrm{i}a_{2'})\Bigg)\Bigg)|0\rangle_{1'}|0\rangle_{2'} \tag{4.3}$$

$$\begin{aligned}\left|e^{i\theta}\right\rangle &= \frac{1}{\sqrt{2}}\Bigg(\exp\left(\frac{\alpha}{\sqrt{2}}(e^{i\theta}a_{1'}^\dagger + ia_{2'}^\dagger) - \frac{\alpha^*}{\sqrt{2}}(e^{-i\theta}a_{1'} - ia_{2'})\right)\\ &\quad + \exp\left(\frac{-\alpha}{\sqrt{2}}(e^{i\theta}a_{1'}^\dagger + ia_{2'}^\dagger) - \frac{-\alpha^*}{\sqrt{2}}(e^{-i\theta}a_{1'} - ia_{2'})\right)\Bigg)|0\rangle_{1'}|0\rangle_{2'}\\ |BS2\rangle &= \frac{1}{\sqrt{2}}\Bigg(\exp\left(\frac{\alpha}{2}(e^{i\theta}(a_1^\dagger + ia_2^\dagger) + i(ia_1^\dagger + a_2^\dagger)) - \frac{\alpha^*}{2}(e^{-i\theta}(a_1 - ia_2) - i(-ia_1 + a_2))\right)\\ &\quad + \exp\left(\frac{-\alpha}{2}(e^{i\theta}(a_1^\dagger + ia_2^\dagger) + i(ia_1^\dagger + a_2^\dagger)) - \frac{-\alpha^*}{2}(e^{-i\theta}(a_1 - ia_2) - i(-ia_1 + a_2))\right)\Bigg)|0\rangle_1|0\rangle_2\\ &= \frac{1}{\sqrt{2}}\Bigg(\underbrace{\exp\left(\frac{\alpha}{2}(e^{i\theta}-1)a_1^\dagger - \frac{\alpha^*}{2}(e^{-i\theta}-1)a_1\right)}_{|\frac{\alpha}{2}(e^{i\theta}-1)\rangle_1}\underbrace{\exp\left(\frac{i\alpha}{2}(e^{i\theta}+1)a_1^\dagger - \frac{-i\alpha^*}{2}(e^{-i\theta}+1)a_1\right)}_{|\frac{i\alpha}{2}(e^{i\theta}+1)\rangle_2}\\ &\quad + \underbrace{\exp\left(\frac{-\alpha}{2}(e^{i\theta}-1)a_1^\dagger - \frac{-\alpha^*}{2}(e^{-i\theta}-1)a_1\right)}_{|-\frac{\alpha}{2}(e^{i\theta}-1)\rangle_1}\underbrace{\exp\left(\frac{-i\alpha}{2}(e^{i\theta}+1)a_1^\dagger - \frac{i\alpha^*}{2}(e^{-i\theta}+1)a_1\right)}_{|-\frac{i\alpha}{2}(e^{i\theta}+1)\rangle_2}\Bigg)|0\rangle_1|0\rangle_2\\ &= \frac{1}{\sqrt{2}}\left(\left|\frac{\alpha}{2}\left(e^{i\theta}-1\right)\right\rangle_1\left|\frac{i\alpha}{2}\left(e^{i\theta}+1\right)\right\rangle_2 + \left|-\frac{\alpha}{2}\left(e^{i\theta}-1\right)\right\rangle_1\left|-\frac{i\alpha}{2}\left(e^{i\theta}+1\right)\right\rangle_2\right) \end{aligned} \tag{4.4}$$

The expectation value of the difference operator $\hat{O} = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$ is now given by

$$\begin{aligned}\langle\hat{O}\rangle &= \frac{1}{2}\left(\left|\frac{i\alpha}{2}\left(e^{i\theta}+1\right)\right|^2 - \left|\frac{\alpha}{2}\left(e^{i\theta}-1\right)\right|^2\right)\\ &= \frac{1}{2}\left(\frac{|\alpha|^2}{4}(\cancel{2} + 2\cos\theta) - \frac{|\alpha|^2}{4}(\cancel{2} - 2\cos\theta)\right)\\ &= \frac{|\alpha|^2}{2}\cos\theta. \end{aligned} \tag{4.5}$$

4.2 Hong-Ou-Mandel interference

Consider a HOM experiment with two indistinguishable photons - one impinging in each input port of a realistic unbalanced BS that has only $|r|^2 = 0.4$. Calculate the maximum HOM-visibility that can be expected in such experimental setting (take into account the $\pi/2$ phase jump for reflection in the BS).

Solution

For the HOM-visibility we need to measure the maximum $N_{c,\text{max}}$ and minimum $N_{c,\text{min}}$ number of coincidences. In general, the number of coincidences (or the probability of coincidence) is given by the second order correlation function

$$\begin{aligned} g^{(2)}(\tau) &= \left\langle \Psi_{\text{out}} | \hat{E}_1^{(-)}(t) \hat{E}_2^{(-)}(t+\tau) \hat{E}_2^{(+)}(t+\tau) \hat{E}_1^{(+)}(t) | \Psi_{\text{out}} \right\rangle \quad \text{where} \quad \hat{E}_2^{(-)}(t) = \mathrm{i}\xi \hat{a}(t), \\ &\sim \left\langle \Psi_{\text{out}} | \hat{a}_1^\dagger(t) \hat{a}_2^\dagger(t+\tau) \hat{a}_2(t+\tau) \hat{a}_2(t) | \Psi_{\text{out}} \right\rangle \\ &= \left\langle \Psi_{\text{out}} | \hat{a}_1^\dagger \mathrm{e}^{\mathrm{i}\omega_1 t} \hat{a}_2^\dagger \mathrm{e}^{\mathrm{i}\omega_2(t+\tau)} \hat{a}_2 \mathrm{e}^{-\mathrm{i}\omega_2(t+\tau)} \hat{a}_1 \mathrm{e}^{-\mathrm{i}\omega_1 t} | \Psi_{\text{out}} \right\rangle \\ &= \left\langle \Psi_{\text{out}} | \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 | \Psi_{\text{out}} \right\rangle. \end{aligned} \tag{4.6}$$

We must also determine the output state $|\Psi_{\text{out}}\rangle$ after the beam splitters. The input state before the beam splitters can be written as

$$|\Psi_{\text{in}}\rangle\, \hat{a}_{1'}^\dagger, \hat{a}_{2'}^\dagger\, |0\rangle\,. \tag{4.7}$$

After the beam splitter the ladder operators transform as follows:

$$\begin{aligned} \hat{a}_1^\dagger &= t\hat{a}_{1'}^\dagger + \mathrm{i}r\hat{a}_{2'}^\dagger \\ \hat{a}_2^\dagger &= t\hat{a}_{2'}^\dagger + \mathrm{i}r\hat{a}_{1'}^\dagger. \end{aligned} \tag{4.8}$$

We can insert these transformations into $|\Psi_{\text{in}}\rangle$ to find the output state

$$\begin{aligned} |\Psi_{\text{out}}\rangle &= (t\hat{a}_{1'}^\dagger + \mathrm{i}r\hat{a}_{2'}^\dagger)(t\hat{a}_{2'}^\dagger + \mathrm{i}r\hat{a}_{1'}^\dagger)\,|0\rangle \\ &= \left[(|t|^2 - |r|^2)\hat{a}_1^\dagger \hat{a}_2^\dagger + \mathrm{i}tr((\hat{a}_1^\dagger)^2 + (\hat{a}_2^\dagger)^2)\right]|0\rangle \\ &= (T-R)\,|11\rangle + \mathrm{i}tr(|20\rangle + |02\rangle). \end{aligned} \tag{4.9}$$

The second term of the equation vanishes in the expectation value (**??**). In the end we just have

$$\begin{aligned} N_{c,\text{min}}(\tau) \sim g^{(2)}(\tau=0) &= \left\langle \Psi_{\text{out}} | \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 | \Psi_{\text{out}} \right\rangle \\ &= (T-R)^2 \left\langle 1\,1 | \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 | 1\,1 \right\rangle \\ &= (T-R)^2 \langle 0\,0|0\,0\rangle = (T-R)^2 = 0.04. \end{aligned} \tag{4.10}$$

On the other hand, the maximum probability of a coincidence measurement is the sum of the two possibility of coincidence measurement, where in case 1 both photons are reflected ($\sim R^2$) and in the second case both photons are transmitted ($\sim T^2$). Here we obtain

$$N_{c,\max} = T^2 + R^2 = 0.52. \tag{4.11}$$

For the visibility we only have to calculate

$$V = \frac{N_{c,\max} - N_{c,\min}}{N_{c,\max} + N_{c,\min}} = \frac{0.48}{0.56} = 0.86. \tag{4.12}$$

For a 50/50 beam splitter we would get $N_{c,\min} = 0$ and thus $V = 1$ as expected.

5 Quantum experiments

5.1 The quantum eraser

We consider the following setup:

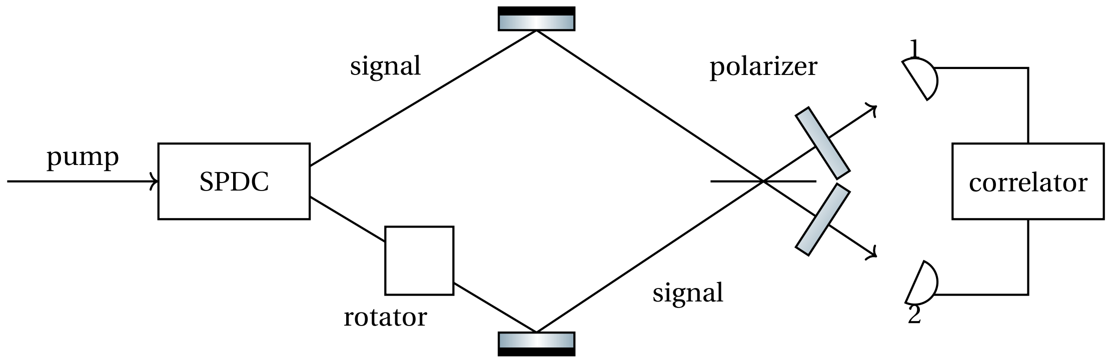

Fig. 2: Signal and idler photons produced by a type-I phase matching SPDC process at DC are aligned to meet at a beamsplitter (dashed line). The rotator (half wave plate) in the idler path allows to retrieve the path information destroying the quantum interference exhibited in the Hong-Ou-Mandel experiment. The polarizers placed before the detectors restore the quantum interference. Ref.: Gerry, Christopher C. and P.L. Knight. "Introductory quantum optics" (2004).

a) *Derive the output state of the interferometer as a function of the rotation angle θ (angle with respect to the horizontal axis). Will the indistinguishably of the single photons be preserved?*

b) *Suppose now the case $\theta = \pi/2$ and that a linear polarizer is placed at an angle θ_1 (with respect to the horizontal axis) in front of the detector in the upper arm and a second polarizer will be placed at an angle θ_2 in front of the lower detector. Calculate the probability of coincident detection of a photon pair depending on θ_1 and θ_2.*

a.) Solution

In the SPDC process one pump photon is used to produce a signal and an idler photon. We assume a type-I phase matching process, i. e. the input state behind the SPDC crystal can be written as

$$|\Psi_{\text{in}}\rangle = |H\rangle_s |H\rangle_i = \hat{a}^\dagger_{s,H}\hat{a}^\dagger_{i,H}|0\rangle\,. \tag{5.1}$$

After the rotator the polarization of the idler photons is rotated by an angle θ. The state after rotation can be expressed as

$$\begin{aligned}|\Psi_{\text{rot}}\rangle &= |H\rangle_s (\cos\theta\,|H\rangle_i + \sin\theta\,|V\rangle_i) \\ &= \cos\theta\,\hat{a}^\dagger_{s,H}\hat{a}^\dagger_{i,H}|0\rangle + \sin\theta\,\hat{a}^\dagger_{s,H}\hat{a}^\dagger_{i,V}|0\rangle\,.\end{aligned} \tag{5.2}$$

Now, using the known transformations for a 50/50 beam splitter we find for the modes 1 and 2 of the detectors

$$\hat{a}_s^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_2^\dagger + \mathrm{i}\hat{a}_1^\dagger) \qquad \hat{a}_i^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \mathrm{i}\hat{a}_2^\dagger). \tag{5.3}$$

We can substitute these transformations into (5.2) to find

$$\begin{aligned}
|\Psi_{\text{out}}\rangle &= \frac{\cos\theta}{2}(\hat{a}_{2,H}^\dagger + \mathrm{i}\hat{a}_{1,H}^\dagger)(\hat{a}_{1,H}^\dagger + \mathrm{i}\hat{a}_{2,H}^\dagger)\,|0\rangle \\
&+ \frac{\sin\theta}{2}(\hat{a}_{2,H}^\dagger + \mathrm{i}\hat{a}_{1,H}^\dagger)(\hat{a}_{1,V}^\dagger + \mathrm{i}\hat{a}_{2,V}^\dagger)\,|0\rangle \\
&= \frac{\cos\theta}{2}(\cancel{\hat{a}_{2,H}^\dagger\hat{a}_{1,H}^\dagger} + \mathrm{i}(\hat{a}_{1,H}^\dagger)^2 + \mathrm{i}(\hat{a}_{2,H}^\dagger)^2 - \cancel{\hat{a}_{1,H}^\dagger\hat{a}_{2,H}^\dagger})\,|0\rangle \\
&+ \frac{\sin\theta}{2}(\hat{a}_{2,H}^\dagger\hat{a}_{1,V}^\dagger + \mathrm{i}\hat{a}_{1,H}^\dagger\hat{a}_{1,V}^\dagger + \mathrm{i}\hat{a}_{2,H}^\dagger\hat{a}_{2,V}^\dagger - \hat{a}_{1,H}^\dagger\hat{a}_{2,V}^\dagger)\,|0\rangle \\
&= \frac{\mathrm{i}\cos\theta}{2}(|HH\rangle_1\,|0\rangle_2 + |0\rangle_1\,|HH\rangle_2) \\
&+ \frac{\sin\theta}{2}(|V\rangle_1\,|H\rangle_2 + \mathrm{i}\,|HV\rangle_1\,|0\rangle_2 + \mathrm{i}\,|0\rangle_1\,|HV\rangle_2 - |H\rangle_1\,|V\rangle_2).
\end{aligned} \tag{5.4}$$

We observe that for a nonzero angle θ the indistinguishably of the single photons will not be preferred, because the blue terms in equation (5.4) will lead to photon coincidences, since here the two photons will be in different spatial modes thus leading to a click in both detectors at the same time.

b.) Solution:

For $\theta = \pi/2$ equation (5.4) will simplify to

$$\left|\Psi_{\text{out}}(\theta = \frac{\pi}{2})\right\rangle = \frac{1}{2}(|V\rangle_1\,|H\rangle_2 - |H\rangle_1\,|V\rangle_2) \tag{5.5}$$

where we neglect the imaginary terms since they will not contribute to coincidences anyway. For the number of coincidences we take the scalar product of the states of the polarizers with our output state. Lets start with the polarizers

$$\begin{aligned}
|\theta_1\rangle &= \sin\theta_1\,|V\rangle_1 + \cos\theta_1\,|H\rangle_1 \\
|\theta_2\rangle &= \sin\theta_2\,|V\rangle_2 + \cos\theta_2\,|H\rangle_2\,.
\end{aligned} \tag{5.6}$$

Now we simply take the scalar product, where we perform this calculation step by step:

$$\begin{aligned}
\langle\theta_1|\Psi_{\text{out}}\rangle &= \frac{1}{2}(\sin\theta_1\,\langle V|_1 + \cos\theta_1\,\langle H|_1)(|V\rangle_1\,|H\rangle_2 - |H\rangle_1\,|V\rangle_2) \\
&= \frac{1}{2}(\sin\theta_1\,|H\rangle_2 - \cos\theta_1\,|V\rangle_2), \\
\Rightarrow \langle\theta_2|\,\langle\theta_1|\Psi_{\text{out}}\rangle &= \frac{1}{2}(\sin\theta_2\,\langle V|_2 + \cos\theta_2\,\langle H|_2)(\sin\theta_1\,|H\rangle_2 - \cos\theta_1\,|V\rangle_2) \\
&= \frac{1}{2}(\cos\theta_2\sin\theta_1 - \sin\theta_2\cos\theta_1) = \frac{1}{2}\sin(\theta_1 - \theta_2).
\end{aligned} \tag{5.7}$$

The probability of coincidence detection is then

$$N_c = |\,\langle\theta_2|\,\langle\theta_1|\Psi_{\text{out}}\rangle\,|^2 = \frac{1}{4}\sin^2(\theta_1 - \theta_2). \tag{5.8}$$

5.2 Optical coherence tomography

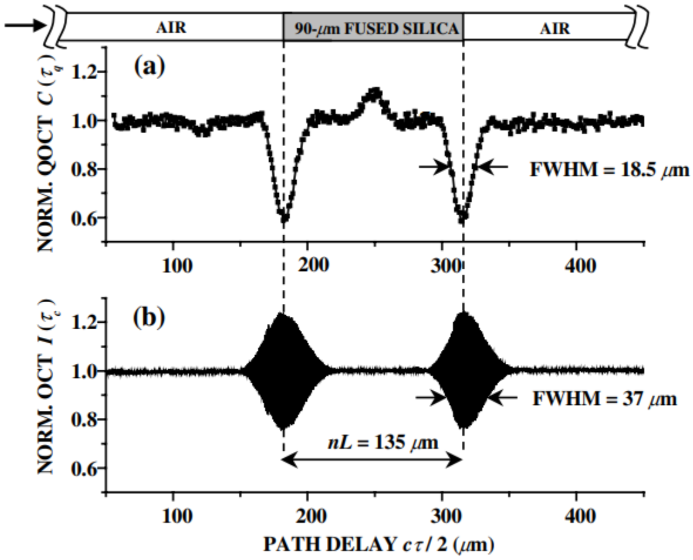

Fig. 3: QOCT and OCT normalized interferograms for a 90 µm fused-silica window in air. Ref.: Phys. Rev. Lett. 91, 083601

Briefly explain how do the signals shown in figure 3 arise, the measurements that are carried out and the physical meaning of the dips and humps in the upper and lower plots, respectively. What is the physical interpretation of the FWHM in both cases.

Solution

The first plot is obtained by performing a measurement of quantum optical coherence. An entangled photon pair (degenerate frequencies) is generated by a nonlinear crystal. One of the photons is sent (normal incidence) to the sample, while the other one is traveling through a delay stage. After that both photons are directed to a beam splitter and the two output modes are detected by two detectors respectively. The interferogram displays the coincidence rate for different path delays of the second entangled photons.

The second plot is obtained by making a classical optical coherence measurement. A photon beam is sent to a beam splitter in a Michelson interferometer setup. In one arm of the interferometer lies the sample acting as the end mirror, while the other arm consists of a movable mirror where we can adjust the delay. The interferogram is then measured by a single detector.

For the first plot the dips correspond to a two-photon interference where the coincidence rate drops to 50 % because the beam of one arm interferes with the reflected beam on the surface/back of the fused-silica window with same reflectivity at the front and back. Thus we observe two dips for surface and back reflection. The hump in the middle of the spectrum corresponds to interference of the two reflections on the surface and back with each other. The FWHM of the dips corresponds to half of the bandwidth of the photon source (and correspondingly of the entangled photons). Due to the nature of entanglement, the bandwidth is smaller by a factor of two than what would expect in the classical case.

For the second plot we observe interference fringes of the reflected photons of one arm with the delayed reference as one would expect for a Michelson interferometer. The FWHM of the interference fringes corresponds to the total bandwidth of the photon source.