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Particles and Fields

All Exercises

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1 First Exercise sheet

Exercise 1: *Use the relation* $\hbar c \approx 197\,\text{MeV}\,\text{fm}$ *valid in SI units to compute your body height in inverse eV for those units where* $\hbar = 1 = c$.

Solution: Because of $1\,\text{MeV} = 10^6\,\text{eV}$ and $1\,\text{fm} = 10^{-15}\,\text{m}$, the product $\hbar c$ can be written as

$$\begin{aligned}\hbar c &= 1.97 \cdot 10^9 \cdot 10^{-15}\,\text{eV}\,\text{m} \equiv 1. \\ \Rightarrow 1\,\text{m} &= 0{,}5 \cdot 10^7\,\frac{1}{\text{eV}}. \end{aligned} \tag{1.1}$$

For a bodyheight of 1,93 m this this leads to a result of $0{,}98 \cdot 10^7\,\frac{1}{\text{eV}}$. The natural units are actually very useful, because they relate all SI units with each other. Let [L] be the dimension of length, [T] the dimension of time and [M] the dimension of mass. Then we see that

$$\begin{aligned}[c] = 1 = [L] \cdot [T]^{-1} \quad &\Rightarrow [L] = [T] = \frac{1}{\text{eV}} \\ [\hbar] = 1 = [M] \cdot [L]^2 \cdot [T]^{-1} \quad &\Rightarrow [M] = [L]^{-1} = \text{eV}\end{aligned}$$

Exercise 2: *Show that the particular Lorentz transformation* Λ *discussed in the lecture, corresponding to a boost along the* x *axis, can be written as* $\exp(-\zeta K_1)$, *where*

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \zeta = \mathit{artanh}(\beta) \qquad \Rightarrow \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.2}$$

Convince yourself that a boost in ageneral direction given by the relative velocity vector $\boldsymbol{\beta}$ *can be written as* $\exp(-\boldsymbol{\zeta} \cdot \boldsymbol{K})$. *Work out the relation between* $\boldsymbol{\beta}$ *and* $\boldsymbol{\zeta}$ *as well as the form of the matrices* K_2 *and* K_3.

Solution: Lets use the TAYLOR series expansion to express the exponential function:

$$\begin{aligned}\exp(-\zeta K_1) &= \mathbb{1} + \sum_{n=1}^{\infty} \frac{(-\zeta K_1)^n}{n!} \\ &= \mathbb{1} - \zeta K_1 + \frac{1}{2}(\zeta K_1)^2 - \frac{1}{3!}(\zeta K_1)^3 + \ldots \\ &= [\mathbb{1} + \frac{1}{2}(\zeta K_1)^2 + \frac{1}{4!}(\zeta K_1)^4] - [\zeta K_1 + \frac{1}{3!}(\zeta K_1)^3 + \frac{1}{5!}(\zeta K_1)^5] \\ &= [\mathbb{1} + \frac{1}{2}\zeta^2 + \frac{1}{4!}\zeta^4]\mathbb{1} - [\zeta + \frac{1}{3!}\zeta^3 + \frac{1}{5!}\zeta^5]K_1. \end{aligned} \tag{1.3}$$

In the last step we used the idempotenz of $(K_1)^2 = \mathbb{1}$. The result of (1.3) can be compared with the series expansion of $\cosh(x)$ and $\sinh(x)$:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots \tag{1.4}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots. \tag{1.5}$$

This results in

$$\exp(-\zeta K_1) = \cosh\zeta \cdot \mathbb{1} - \sinh\zeta \cdot K_1. \tag{1.6}$$

This result is compared to Λ. If we look at the upper left corner of the matrix we can see that

$$\begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\zeta & -\sinh\zeta \\ -\sinh\zeta & \cosh\zeta \end{pmatrix}. \tag{1.7}$$

By comparing the elements of both matrices we find

$$\gamma = \cosh\zeta, \quad \gamma\beta = \sinh\zeta \qquad \Rightarrow \beta = \tanh\zeta. \tag{1.8}$$

The other two matrices K_2 and K_3 the describing matrices for a LORENTZ boost in y- and z-direction, which can be written in the following way

$$y-\text{direction} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Rightarrow \Lambda = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.9}$$

$$z-\text{direction} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \Rightarrow \Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}. \tag{1.10}$$

For a boost in a general direction $\boldsymbol{v} = (v_1, v_2, v_3) \Rightarrow \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)$ the rapidity $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)$ and the K-matrices $\boldsymbol{K} = (K_1, K_2, K_3)$ also become vectors. In fact, the vector of rapidity $\boldsymbol{\zeta}$ points in β-direction. The vectors can be normalized

$$\hat{\boldsymbol{\zeta}} = \frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|} = \hat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} \qquad |\hat{\boldsymbol{\zeta}}| = \operatorname{artanh}(|\hat{\boldsymbol{\beta}}|). \tag{1.11}$$

The LORENTZ transformation Λ can now be written as

$$\Lambda = \exp(-\boldsymbol{\zeta} \cdot \boldsymbol{K}) = \exp\left(-\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}\operatorname{artanh}(\beta)\right). \tag{1.12}$$

Exercise 3: *Verify that the matrix Λ given above satisfies the relation*

$$g_{\mu\nu} = g_{\kappa\lambda}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu} \tag{1.13}$$

where the metric is $g = diag(1,-1,-1,-1)$.

Solution: Lets first solve the task by using the matrix notation of the LORENTZ transformation. Because of $(\Lambda^T)^{\nu}_{\mu} = \Lambda^{\nu}_{\mu}$ the right hand side of (1.13) can be written as

$$(\Lambda^T)^{\kappa}_{\mu}\underbrace{g_{\kappa\lambda}\Lambda^{\lambda}_{\nu}}_{\Theta_{\kappa\nu}}, \tag{1.14}$$

where $\Theta_{\kappa\nu}$ can be expressed as a matrix multiplication

$$\Theta_{\kappa\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.15}$$

Then the whole matrix multiplication can be performed by multiplying the transposed LORENTZ transformation on the left to $\Theta_{\kappa\nu}$

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Theta_{\kappa\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.16}$$

Another way to show this relation is to perform the summation of the coefficients of the LORENTZ transformation

$$g_{\kappa\lambda}\Lambda^{\kappa}_{\mu}\Lambda^{\lambda}_{\nu} = g_{00}\Lambda^{0}_{\mu}\Lambda^{0}_{\nu} + g_{11}\Lambda^{1}_{\mu}\Lambda^{1}_{\nu} + g_{22}\Lambda^{2}_{\mu}\Lambda^{2}_{\nu} + g_{33}\Lambda^{3}_{\mu}\Lambda^{3}_{\nu}. \tag{1.17}$$

The indices μ, ν can be chosen freely ($\mu, \nu = 0, 1, 2, 3$). The sums can be calculated seperatly:

$$\begin{aligned}
&\sum_{00} = (\Lambda^0_0)^2 - (\Lambda^1_0)^2 = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2) = 1 \\
&\sum_{01} = \sum_{02} = \sum_{03} = 0 \\
&\sum_{11} = -1, \quad \sum_{11} = \sum_{12} = \sum_{13} = 0 \\
&\sum_{22} = -1, \quad \sum_{21} = \sum_{22} = \sum_{23} = 0 \\
&\sum_{33} = -1, \quad \sum_{31} = \sum_{32} = \sum_{33} = 0.
\end{aligned}$$

This actually results in $g_{\mu\nu}$.

2 Second Exercise sheet

In general the Euler-Lagrange equations of motion for a generic action function $S[\Phi]$, given by

$$S[\Phi] = \int \mathrm{d}^4 x \mathscr{L}(\Phi, \partial\mu\Phi). \tag{2.1}$$

where $\mathscr{L}(\Phi, \partial\mu\Phi)$: Lagrangian density which describes the field theory, are given by

$$\left(\frac{\partial \mathscr{L}}{\partial \Phi}\right) - \partial\mu\left(\frac{\partial \mathscr{L}}{\partial\partial\mu\Phi}\right) = 0. \tag{2.2}$$

Here $\Phi(x^\mu) = \Phi(t, \boldsymbol{x})$ and $\partial\mu = \frac{\partial}{\partial x^\mu}$.

Exercise 3: *Use the Euler-Lagrange equations to derive the equations of motion for*

a) *Maxwells electrodynamics,*

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu. \tag{2.3}$$

b) *The theory of a complex Klein-Gordon field,*

$$\mathscr{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi, \tag{2.4}$$

where $\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2), \phi_{1,2} \in \mathbf{R}$. Show that the equations of motion can also more conventiently be obtained if ϕ and ϕ^* are considered as independent fields.

c) *Schrödinger theory,*

$$\mathscr{L} = \psi^* \mathrm{i}\partial_t \psi - \frac{1}{2m} (\vec{\boldsymbol{\nabla}}\psi^*) \cdot (\vec{\boldsymbol{\nabla}}\psi) - V(\boldsymbol{x})\psi^*\psi. \tag{2.5}$$

Use the same trick as in b.) and consider ψ and ψ^* as independent.

a.) Solution: For the derivation of the equations of motion we use the general relation

$$\frac{\partial \partial_\alpha A_\beta}{\partial \partial_\gamma A_\eta} = \delta^\gamma_\alpha \delta^\eta_\beta. \tag{2.6}$$

Now we can write the Euler-Lagrange equations as

$$\frac{\partial \mathscr{L}}{\partial A_\mu} = \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\nu)} \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.7}$$

First we rewrite the Lagrange density in terms of the generalized vector potential (assuming a flat space time with $\eta^{\alpha\beta} = g^{\alpha\beta}$)

$$\mathscr{L} = -\frac{1}{4} g^{\alpha\mu} g^{\beta\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha). \tag{2.8}$$

At first we compute the derivative of the lagrange density with respect to $\partial_\kappa A_\lambda$

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial(\partial_\kappa A_\lambda)} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}\frac{\partial}{\partial(\partial_\kappa A_\lambda)}[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\alpha A_\beta - \partial_\beta A_\alpha)] \\ &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}[(\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{\partial}{\partial(\partial_\kappa A_\lambda)}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \\ &\quad + (\partial_\alpha A_\beta - \partial_\beta A_\alpha)\frac{\partial}{\partial(\partial_\kappa A_\lambda)}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)].\end{aligned} \tag{2.9}$$

We can now use equation (2.6) for the derivatives

$$\frac{\partial}{\partial(\partial_\kappa A_\lambda)}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = \delta^\kappa_\alpha\delta^\lambda_\beta - \delta^\kappa_\beta\delta^\lambda_\alpha, \tag{2.10}$$

which leads to

$$\begin{aligned}\frac{\partial\mathscr{L}}{\partial(\partial_\kappa A_\lambda)} &= -\frac{1}{4}\eta^{\alpha\mu}\eta^{\beta\nu}[\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{=F_{\mu\nu}}(\delta^\kappa_\alpha\delta^\lambda_\beta - \delta^\kappa_\beta\delta^\lambda_\alpha) + \underbrace{(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}_{=F_{\alpha\beta}}(\delta^\kappa_\mu\delta^\lambda_\nu - \delta^\kappa_\nu\delta^\lambda_\mu)] \\ &= -\frac{1}{4}\left[\left(\eta^{\kappa\mu}\eta^{\lambda\nu} - \eta^{\lambda\mu}\eta^{\kappa\nu}\right)F_{\mu\nu} + \left(\eta^{\kappa\alpha}\eta^{\lambda\beta} - \eta^{\lambda\alpha}\eta^{\kappa\beta}\right)F_{\alpha\beta}\right] \\ &= -\frac{1}{4}(F^{\kappa\lambda} - F^{\lambda\kappa} + F^{\kappa\lambda} - F^{\lambda\kappa}) \\ &= -F^{\kappa\lambda} \\ \Rightarrow \partial_\kappa\frac{\partial\mathscr{L}}{\partial(\partial_\kappa A_\lambda)} &= -\partial_\kappa F^{\kappa\lambda}.\end{aligned} \tag{2.11}$$

By using that $\partial\mathscr{L}/\partial(A_\kappa) = -J^\kappa$ we can write the equation of motion as

$$\boxed{J^\kappa = \partial_\kappa F^{\kappa\lambda}}. \tag{2.12}$$

b.) Solution: $\mathscr{L}$ can be rewritten in terms of $\phi_{1,2}$ as

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu\phi_1)(\partial^\mu\phi_1) - \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}(\partial_\mu\phi_2)(\partial^\mu\phi_2) - \frac{1}{2}m^2\phi_2^2 \\ &= (\underbrace{\phi_1, \partial_\mu\phi_1}_{\text{e.o.m.}}, \underbrace{\phi_2, \partial_\mu\phi_2}_{\text{e.o.m.}}).\end{aligned} \tag{2.13}$$

The equations of motion (e.o.m.) read

$$\frac{\partial\mathscr{L}}{\partial\phi_1} - \partial_\nu\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_1)} = 0 \tag{2.14}$$

$$\frac{\partial\mathscr{L}}{\partial\phi_2} - \partial_\nu\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi_2)} = 0. \tag{2.15}$$

As was done in the lecture we get

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_1}=-m^2\phi_1,\quad \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi_2)}&=\frac{1}{2}\frac{\partial}{\partial(\partial_\nu\phi_1)}[(\partial_\mu\phi_1)(\partial^\mu\phi_1)]\\ &=\frac{1}{2}\left[\delta^\nu_\mu(\partial^\mu\phi_1)+(\partial_\mu\phi_1)\frac{\partial}{\partial(\partial_\nu\phi_1)}(\partial^\mu\phi_1)\right]\\ &=\frac{1}{2}\left[\partial^\nu\phi_1+\eta^{\mu\rho}(\partial_\mu\phi_1)\frac{\partial(\partial_\rho\phi_1)}{\partial(\partial_\nu\phi_1)}\right]\\ &=\frac{1}{2}\left[\partial^\nu\phi_1+(\partial^\rho\phi_1)\delta^\nu_\rho\right]\\ &=\frac{1}{2}\left[\partial^\nu\phi_1+\partial^\nu\phi_1\right]=\partial^\nu\phi_1.\end{aligned}\tag{2.16}$$

From both ϕ_1 and ϕ_2 this results in the Klein-Gordon equation

$$(\Box+m^2)\phi_1=0,\qquad(\Box+m^2)\phi_2=0,\quad\text{where }\Box=\partial_\mu\partial^\mu.\tag{2.17}$$

c.) Solution: Here we consider ψ and ψ^* as independent. The covariant differential is

$$\partial_\mu=\frac{\partial}{\partial x^\mu}=\left(\frac{\partial}{\partial t},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right).\tag{2.18}$$

Then, the Euler-Lagrange equations of motion read

$$\frac{\partial\mathcal{L}}{\partial\psi}-\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}-\vec{\nabla}\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\psi)}=0\tag{2.19}$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*}-\vec{\nabla}\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\psi^*)}=0.\tag{2.20}$$

Thus, for equation (2.19) and (2.20) we find

$$\frac{\partial\mathcal{L}}{\partial\psi}=-\psi^*V(\boldsymbol{x}),\quad\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)}=\mathrm{i}\psi^*,\quad\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\psi)}=-\frac{1}{2m}(\vec{\nabla}\psi^*)\tag{2.21}$$

$$\frac{\partial\mathcal{L}}{\partial\psi^*}=-\psi V(\boldsymbol{x})+\mathrm{i}\partial_t\psi,\quad\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\psi^*)}=-\frac{1}{2m}(\vec{\nabla}\psi).\tag{2.22}$$

Hence this leads to

$$\mathrm{i}(\partial_t\Psi^*)=\frac{1}{2m}\Delta\psi^*-\psi^*V(\boldsymbol{x})\tag{2.23}$$

$$\mathrm{i}(\partial_t\Psi)=-\frac{1}{2m}\Delta\psi+\psi V(\boldsymbol{x}).\tag{2.24}$$

Exercise 4: *Consider the following Lagrange density (Proca theory)*

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}\mu^2A_\mu A^\mu-J^\mu A_\mu.\tag{2.25}$$

a) *Derive the equations of motion.*

b) *Which condition hast to be imposed on A_μ in order to maintain current conservation? How does this simplify the equations of motion?*

c) *Consider the static limit, i.e, A_μ becomes independent of time. Let the current be given by a point charge $J_0 = q\delta^{(3)}(\boldsymbol{x}), J_i = 0$. How does the static potential A_0 look like? Interpret the quantity μ in the light of this result.*

a.) Solution Using the Euler-Lagrange equation we get

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial A_\alpha} - \partial_\beta \frac{\partial \mathscr{L}}{\partial(\partial_\beta A_\alpha)} &= 0 \\ \mu^2 A^\alpha - J^\alpha + \partial_\beta F^{\beta\alpha} &= 0 \\ \mu^2 A^\alpha + \partial_\beta F^{\beta\alpha} &= J^\alpha. \end{aligned} \tag{2.26}$$

b.) Solution Normally $\mu = 0$. This leads to a gauge symmetry

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu - \partial_\mu x_i \\ \Rightarrow \mathscr{L}(A'_\mu, \partial_m A'_\mu) &= \mathscr{L}(A_\mu, \partial_m A_\mu) \end{aligned} \tag{2.27}$$

Here we have $\mu \neq 0$. The continuity equation yields

$$\begin{aligned} \partial_\alpha J^\alpha = \partial_t \rho + \vec{\boldsymbol{\nabla}} \cdot \boldsymbol{J} = 0 &= \mu^2 \partial_\alpha A^\alpha + \underbrace{\partial_\alpha \partial_\beta F^{\alpha\beta}}_{0} \\ 0 &= \mu^2 \partial_\alpha A^\alpha. \end{aligned} \tag{2.28}$$

The version of Maxwells equations with mass μ the gauge symmetry is broken. We have to stick to the Lorentz gauge.

c.) Solution: We can use the result of b.)

$$\begin{aligned} \partial_\mu A^\mu = 0 \Rightarrow \partial_\mu F^{\mu\nu} = \partial_\mu[\partial^\mu A^\nu - \partial^\nu A^\mu] \qquad &= \partial_\mu \partial^\mu A^\nu - \underbrace{\partial^\mu \partial_\mu A^\mu}_{=0} \\ = \Box A^\nu. & \end{aligned} \tag{2.29}$$

The equations of motion then read

$$\Box A^\nu + \mu^2 A^\nu = J^\nu. \tag{2.30}$$

Now we use the $J_0 = q\delta^{(3)}(\boldsymbol{x})$. In the static limit we get

$$\begin{aligned} A_\mu = A_\mu(\boldsymbol{x}) \Rightarrow \Box A_\mu &= \Box A_\mu(\boldsymbol{x}) \\ &= \partial_\mu \partial^\mu A_\mu(\boldsymbol{x}) = \nabla^2 A_\mu(\boldsymbol{x}). \end{aligned} \tag{2.31}$$

Therefore we get

$$-\nabla^2 A_\mu + \mu^2 A_\mu = J_\mu. \tag{2.32}$$

The fields generated by a static charge are spherically symmetric. That means $A_0(\boldsymbol{x}) = A_0(r)$. Then we can write the Laplacian in spherical coordinates:

$$\left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2\frac{\partial A_0(r)}{\partial r}\right) - \mu^2 A_0(r) = q\delta(r). \tag{2.33}$$

We assume that the solution is

$$A_0(r) = () \tag{2.34}$$

3 Third Exercise sheet

Exercise 7: *From a pragmatic viewpoint, functioal differentiation can be defined by the conditions that the algebraic rules for standard derivatives apply,*

$$\begin{aligned}\frac{\delta}{\delta\phi(x)}\big(F_1[\phi]+F_2[\phi]\big) &= \frac{\delta}{\delta\phi(x)}F_1[\phi]+\frac{\delta}{\delta\phi(x)}F_2[\phi], \quad (\textit{linearity})\\ \frac{\delta}{\delta\phi(x)}\big(F_1[\phi]F_2[\phi]\big) &= F_1[\phi]\frac{\partial}{\partial\phi(x)}F_2[\phi]+F_2[\phi]\frac{\delta}{\delta\phi(x)}F_1[\phi], \quad (\textit{Leibniz rule})\end{aligned} \tag{3.1}$$

where $F_i[\phi]$ are functionals of ϕ, and that additionally we have:

$$\frac{\delta}{\delta\phi(y)}\phi(x)=\delta^{(D)}(x-y). \tag{3.2}$$

Verify that

$$\begin{aligned}\frac{\delta}{\delta\phi(y)}\int \mathrm{d}^D x\,\phi(x)J(x) &= J(y),\\ \frac{\delta}{\delta\phi(y)}\exp\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right) &= J(y)\exp\left(\int_x \phi(x)J(x)\right).\end{aligned} \tag{3.3}$$

Solution: We can use the Leibniz rule and change integral and derivative to show that

$$\begin{aligned}\frac{\delta}{\delta\phi(y)}\int \mathrm{d}^D x\,\phi(x)J(x) &= \int \mathrm{d}^D x\,\frac{\delta\phi(x)}{\delta\phi(y)}J(x)+\int \mathrm{d}^D x\,\phi(x)\cancel{\frac{\delta J(x)}{\delta\phi(y)}}\\ &= \int \mathrm{d}^D x\,\delta(x-y)J(x)=J(y).\end{aligned} \tag{3.4}$$

The second relation can be shown by rewriting the exponential function as its Taylor series expansion

$$\begin{aligned}\frac{\delta}{\delta\phi(y)}\exp\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right) &= \frac{\delta}{\delta\phi(y)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right)^n\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{\delta}{\delta\phi(y)}\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right)^n\\ &= \sum_{n=1}^{\infty}\frac{1}{n!}n\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right)^{n-1}J(y)\\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right)^{m}J(y)\\ &= \exp\left(\int \mathrm{d}^D x\,\phi(x)J(x)\right)J(y).\end{aligned} \tag{3.5}$$

Exercise 8: *Given a classical action S for a field $\phi(x)$ in spacetime. We can formulate Hamilton's principle with the aid of the functional derivative:*

$$\frac{\delta S[\phi]}{\delta\phi(x)} = 0. \tag{3.6}$$

Show that for actions of the type $S[\phi] = \int \mathrm{d}^D y\, \mathscr{L}(\phi, \partial_\mu\phi; y)$, we can obtain the Euler-Lagrange equations as discussed in the lecture.

Solution: We can use the action principle $\delta S[\phi] = 0$ to obtain the Euler-Lagrange equations

$$\begin{aligned}
0 = \frac{\delta S[\phi]}{\delta\phi(x)} &= \int \mathrm{d}^D y \frac{\delta}{\delta\phi(x)} \mathscr{L}(\phi, \partial_\mu\phi; y) \\
&= \int \mathrm{d}^D y \left[\frac{\delta\phi(y)}{\delta\phi(x)} \frac{\partial\mathscr{L}}{\partial\phi(y)} + \frac{\delta(\partial\mu\phi(y))}{\delta\phi(x)} \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(x))} \right] \\
&= \int \mathrm{d}^D y \left[\delta^{(D)}(y-x) \frac{\partial\mathscr{L}}{\partial\phi(y)} + \partial_\mu \delta^{(D)}(y-x) \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(y))} \right].
\end{aligned} \tag{3.7}$$

We can now use integration by parts to bring the derivative ∂_μ to the second factor

$$= \frac{\partial\mathscr{L}}{\partial\phi(x)} + \underbrace{\int_{\partial V} \mathrm{d}^{D-1} y\, \partial_\mu \delta^{(D)}(y-x) \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(x))}}_{=0} - \int \mathrm{d}^D y\, \delta^{(D)}(y-x) \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(y))}. \tag{3.8}$$

The integral vanishes, because we assume that x is not on the boundary of the integration volume.

$$0 = \frac{\partial\mathscr{L}}{\partial\phi(x)} - \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(x))}. \tag{3.9}$$

Exercise 9: *For a classical field $\phi(\boldsymbol{x}, t)$ with an associated canonical conjugate momentum density $\pi(\boldsymbol{x}, t)$, we can define the Poisson brackets analogously to classical mechanics. Let $A[\phi, \pi]$ and $B[\phi, \pi]$ be two general phase space functionals, then the Poisson bracket in $d = D - 1$ space dimensions is given by (we ignore the time argument t in the following for simplicity)*

$$\{A, B\} := \int \mathrm{d}^d z \left(\frac{\delta A}{\delta\phi(\boldsymbol{z})} \frac{\delta B}{\delta\pi(\boldsymbol{z})} - \frac{\delta A}{\delta\pi(\boldsymbol{z})} \frac{\delta B}{\delta\phi(\boldsymbol{z})} \right). \tag{3.10}$$

a) *Verify the fundamental Poisson brackets*

$$\{\phi(\boldsymbol{x}), \phi(\boldsymbol{y})\} = 0, \quad \{\pi(\boldsymbol{x}), \pi(\boldsymbol{y})\} = 0, \quad \{\phi(\boldsymbol{x}), \pi(\boldsymbol{y})\} = \delta^{(d)}(\boldsymbol{x} - \boldsymbol{y}). \tag{3.11}$$

The time evolution of the field and the momentum is generated by the Hamilton function H according to the canonical equations of motion

$$\dot{\phi}(\boldsymbol{x}) = \{\phi(\boldsymbol{x}), H\}, \quad \dot{\pi}(\boldsymbol{x}) = \{\pi(\boldsymbol{x}), H\}. \tag{3.12}$$

b) *Compute the equations of motion for Klein-Gordon theory with the Hamilton function*

$$H \equiv \int \mathrm{d}^d y \mathscr{H}(\boldsymbol{y}) = \int \mathrm{d}^d y \frac{1}{2}\left(\pi^2 + (\vec{\boldsymbol{\nabla}}\phi)^2 + m^2\phi^2\right) \tag{3.13}$$

where $\mathscr{H}(\boldsymbol{y})$ is the Hamilton density.

a.) Solution:

$$\{\phi(\boldsymbol{x}),\phi(\boldsymbol{y})\} = \int \mathrm{d}^d z \left(\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{z})}\cancel{\frac{\delta\phi(\boldsymbol{y})}{\delta\pi(\boldsymbol{z})}} - \cancel{\frac{\delta\phi(\boldsymbol{x})}{\delta\pi(\boldsymbol{z})}}\frac{\delta\phi(\boldsymbol{y})}{\delta\phi(\boldsymbol{z})}\right) = 0 \tag{3.14}$$

$$\{\pi(\boldsymbol{x}),\pi(\boldsymbol{y})\} = \int \mathrm{d}^d z \left(\cancel{\frac{\delta\pi(\boldsymbol{x})}{\delta\phi(\boldsymbol{z})}}\frac{\delta\pi(\boldsymbol{y})}{\delta\pi(\boldsymbol{z})} - \frac{\delta\pi(\boldsymbol{x})}{\delta\pi(\boldsymbol{z})}\cancel{\frac{\delta\pi(\boldsymbol{y})}{\delta\phi(\boldsymbol{z})}}\right) = 0 \tag{3.15}$$

$$\begin{aligned}\{\phi(\boldsymbol{x}),\pi(\boldsymbol{y})\} &= \int \mathrm{d}^d z \Big(\underbrace{\frac{\delta\phi(\boldsymbol{x})}{\delta\phi(\boldsymbol{z})}}_{=\delta(\boldsymbol{x}-\boldsymbol{z})} \underbrace{\frac{\delta\pi(\boldsymbol{y})}{\delta\pi(\boldsymbol{z})}}_{=\delta(\boldsymbol{y}-\boldsymbol{z})} - \cancel{\frac{\delta\phi(\boldsymbol{x})}{\delta\pi(\boldsymbol{z})}}\cancel{\frac{\delta\pi(\boldsymbol{y})}{\delta\phi(\boldsymbol{z})}}\big)\delta^{(d)}(\boldsymbol{x}-\boldsymbol{y}\Big) \\ &= \int \mathrm{d}^d z \delta(\boldsymbol{x}-\boldsymbol{z})\delta(\boldsymbol{y}-\boldsymbol{z}) = \delta(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{3.16}$$

b.) Solution:

First we introduce some useful relations concerning Poisson brackets:

$$\{A,BC\} = \{A,B\}C + B\{A,C\} \tag{3.17}$$

$$\{A,B^2\} = 2B\{A,B\}. \tag{3.18}$$

Lets first calculate the time derivative of $\phi(\boldsymbol{x})$

$$\begin{aligned}\dot{\phi}(\boldsymbol{x}) = \{\phi(\boldsymbol{x}),H\} &= \int \mathrm{d}^d y \left\{\phi(\boldsymbol{x}),\frac{1}{2}\pi^2(\boldsymbol{y})\right\} \\ &\overset{(3.18)}{=} \int \mathrm{d}^d y 2\pi(\boldsymbol{y})\frac{1}{2}\{\phi(\boldsymbol{x}),\pi(\boldsymbol{y})\} \\ &= \int \mathrm{d}^d y \pi(\boldsymbol{y})\delta^{(D)}(\boldsymbol{x}-\boldsymbol{y}) = \pi(\boldsymbol{x}).\end{aligned} \tag{3.19}$$

We can compute the second derivative of $\phi(\boldsymbol{x})$ by deriving the first derivative of $\pi(\boldsymbol{x})$

$$\begin{aligned}\dot{\pi}(\boldsymbol{x}) = \{\pi(\boldsymbol{x}),H\} &= \int \mathrm{d}^d y \left\{\pi(\boldsymbol{x}),\frac{1}{2}(\vec{\boldsymbol{\nabla}}\phi(\boldsymbol{y}))^2 + \frac{1}{2}m^2\phi(\boldsymbol{y})\right\} \\ &\overset{(3.18)}{=} \int \mathrm{d}^d y \left[\frac{1}{2}\{\pi(\boldsymbol{x}),\partial_i\phi(\boldsymbol{y})\}\big(2\partial_i\phi(\boldsymbol{y})\big) + \frac{1}{2}\{\pi(\boldsymbol{x}),\phi(\boldsymbol{y})\}2m^2\phi(\boldsymbol{y})\right]\end{aligned} \tag{3.20}$$

We can pull the derivative out of the Poisson bracket and use its anti-commutating properties $\{A,B\} = -\{B,A\}$

$$= \int \mathrm{d}^d y \left[-\partial_i\delta^{(D)}(\boldsymbol{x}-\boldsymbol{y})\partial_i\phi(\boldsymbol{y}) - m^2\delta^{(D)}(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y})\right]. \tag{3.21}$$

We can again perform an integration by parts and transfer the derivative of the Delta-function to the second factor

$$\begin{aligned}\dot{\pi}(\boldsymbol{x}) &= \int \mathrm{d}^d y \left[+\delta^{(D)}(\boldsymbol{x}-\boldsymbol{y})(\partial_i \partial_i \phi(\boldsymbol{y})) - m^2 \delta^{(D)}(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y})\right] \\ &= [(\partial_i)^2 - m^2]\phi(\boldsymbol{x}).\end{aligned} \tag{3.22}$$

Now we can express the second time derivative of the field $\phi(\boldsymbol{x})$:

$$\begin{aligned}\ddot{\phi}(\boldsymbol{x}) &= [(\partial_i)^2 - m^2]\phi(\boldsymbol{x}) \\ \Rightarrow [(\partial_t)^2 - (\partial_i)^2 + m^2]\phi &= 0 \\ \Rightarrow [\Box + m^2]\phi &= 0.\end{aligned} \tag{3.23}$$

4 Fourth Exercise sheet

Exercise 10: *Consider a classical field theory which is invariant under translations $x_\mu \to x_\mu - a_\mu$. The corresponding Noether charge is given by the 4-momentum of the field ϕ,*

$$P^\mu = \int \mathrm{d}^3x\,(\Pi\partial^\mu\phi - g^{\mu 0}\mathscr{L}). \tag{4.1}$$

Show that the Noether charge is also the generator of spacetime translations, i. e., show that

$$\delta\phi = -a_\mu\{\phi(\boldsymbol{x}), P^\mu\}. \tag{4.2}$$

Solution: From the lecture notes we deduce that under infinitesimal spacetime translations

$$\begin{aligned} x_\mu \to x_\mu - a_\mu, \quad \phi(x) \to \phi'(x) &= \phi(x-a) = \phi(x) - a_\mu\partial^\mu\phi(x) \\ &= \phi(x) + \delta\phi(x) \quad \text{with} \quad \delta\phi(x) = -a_\mu\partial^\mu\phi(x). \end{aligned} \tag{4.3}$$

First, let us look at the 0th component of the Noether charge

$$P^0 = \int \mathrm{d}^3x\,(\Pi\dot\phi - \mathscr{L}) = \int \mathrm{d}^3x\,\mathscr{H} = H \tag{4.4}$$

From Hamiltons equations of motion we obtain the evolution equation generated by P^0 as

$$\dot\phi = \partial^0\phi = \{\phi, H\} = \{\phi, P^0\}. \tag{4.5}$$

For the spatial components ($i = 1,2,3$) we use $g^{i0} = 0$ to find

$$P^i = \int \mathrm{d}^3x\,\Pi\partial^i\phi. \tag{4.6}$$

Then, the Poisson bracket evaluates as

$$\begin{aligned} \{\phi, P^i\} &= \int \mathrm{d}^3y\,\{\phi(x), \Pi(y)\partial^i\phi(y)\} \quad \text{use} \quad \{A, BC\} = B\{A,C\} + \{A,B\}C \\ &= \int \mathrm{d}^3y\,\underbrace{\{\phi(x), \Pi(y)\}}_{\delta(x-y)}\partial^i\phi(y) + \underbrace{\{\phi(x), \partial^i\phi(y)\}}_{=0}\Pi(y) = \partial^i\phi(x). \end{aligned} \tag{4.7}$$

Now, combining equations (4.5) and (4.7) leads to

$$\{\phi, P^\mu\} = \partial^\mu\phi(x) \quad \Rightarrow \quad -a_\mu\{\phi(x), P^\mu\} \overset{(4.3)}{=} \delta\phi(x). \tag{4.8}$$

Exercise 11: *use Noether's theorem to construct the (canonical) energy-momentum tensor for the classical electromagnetic field from the Lagrangian*

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.9}$$

Convince youself that this does not result in a symmetric tensor, i. e., $T^{\mu\nu} \neq T^{\nu\mu}$; also the result is not gauge invariant. A symmetric gauge-invariant tensor can nevertheless be constructed by adding a term of the form

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}, \quad \text{where} \quad K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}. \tag{4.10}$$

The anti-symmetry of $K^{\lambda\mu\nu}$ with respect to its first two indices guarantees that Noether's conservation law still applies, $\partial_\mu T^{\mu\nu} = 0$. Show that this construction with $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$ leads to the desired (symmetric, gauge-invariant) result. Show also the components are related to standard quantities, such as the energy density $\varepsilon = T^{00} = \frac{1}{2}(E^2 + B^2)$ and the Poynting vector (momentum density) $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$, where $S^i = \hat{T}^{0i}$. (Hint: for the last step, use $E^i = -F^{0,i}$ and $\varepsilon^{ijk}B^k = -F^{ij}$). Also show that $\hat{T}^{\mu\nu}$ is traceless.

Solution: In the lecture we defined the canonical energy-momentum tensor as

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\partial^\nu \phi - g^{\mu\nu}\mathscr{L}. \tag{4.11}$$

Then, using $\phi \to A_\lambda(x)$ we can find the Maxwell energy-momentum tensor

$$\begin{aligned} T^{\mu\nu}_{\text{Maxwell}} &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - g^{\mu\nu}\mathscr{L} \\ &\overset{(2.11)}{=} -F^{\mu\lambda}(\partial^\nu A_\lambda) - g^{\mu\nu}\mathscr{L} \\ &\overset{(4.9)}{=} -g^{\nu\rho}F^{\mu\lambda}(\partial_\rho A_\lambda) + \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} \neq T^{\nu\mu}_{\text{Maxwell}}. \end{aligned} \tag{4.12}$$

Under gauge transformations

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x), \tag{4.13}$$

the energy-momentum tensor changes as

$$T'^{\mu\nu} = -F'^{\mu\lambda}\partial^\nu A'_\lambda + \frac{1}{4}g^{\mu\nu}F'_{\lambda\kappa}F'^{\lambda\kappa}. \tag{4.14}$$

The $F_{\mu\nu}$ tensor is actually gauge invariant

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu\Lambda) - \partial_\nu(A_\mu + \partial_\mu\Lambda) \\ &= \underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}_{F_{\mu\nu}} + \underbrace{\partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda}_{=0}. \end{aligned} \tag{4.15}$$

Then, equation (4.14) can be written as

$$\begin{aligned} T'^{\mu\nu} &= -F^{\mu\lambda}\partial^\nu[A_\lambda + \partial_\lambda\Lambda] + \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} \\ &= \underbrace{\frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} - F^{\mu\lambda}(\partial^\nu A_\lambda)}_{T^{\mu\nu}_{\text{Maxwell}}} - \underbrace{F^{\mu\lambda}(\partial^\nu\partial_\lambda\Lambda)}_{\text{breaks gauge invariance}}. \end{aligned} \tag{4.16}$$

Now, consider the modified energy-momentum tensor from equation (4.10). Note that the following relation holds:

$$\partial_\mu \hat{T}^{\mu\nu} = \underbrace{\partial_\mu T^{\mu\nu}_{\text{Maxwell}}}_{=0} + \underbrace{\partial_\mu\partial_\lambda K^{\lambda\mu\nu}}_{=0} = 0. \tag{4.17}$$

We can show that the second part vanishes by using the anti-symmetry of $K^{\lambda\mu\nu}$

$$\partial_\mu\partial_\lambda K^{\lambda\mu\nu} = \partial_\lambda\partial_\mu K^{\lambda\mu\nu} = \partial_\mu\partial_\lambda K^{\mu\lambda\nu} = -\partial_\mu\partial_\lambda K^{\lambda\mu\nu} = 0. \tag{4.18}$$

If we now choose $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, then

$$\begin{aligned} \hat{T}^{\mu\nu} &= T^{\mu\nu}_{\text{Maxwell}} + \partial_\lambda(F^{\mu\lambda}A^\nu) \\ &\quad \text{with} \quad \partial_\lambda(F^{\mu\lambda}A^\nu) = \underbrace{(\partial_\lambda F^{\mu\lambda})}_{=0}A^\nu + F^{\mu\lambda}(\partial_\lambda A^\nu) \\ &= \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} - F^{\mu\lambda}(\partial^\nu A_\lambda) + F^{\mu\lambda}(\partial_\lambda A^\nu) \\ &= \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} + F^{\mu\lambda}F_\lambda{}^\nu \\ &= \frac{1}{4}g^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa} + g^{\nu\rho}F_{\lambda\rho}F^{\mu\lambda} = \hat{T}^{\nu\mu}. \end{aligned} \tag{4.19}$$

Since $F'_{\mu\nu} = F_{\mu\nu}$ is gauge invariant, $\hat{T}'^{\mu\nu} = \hat{T}^{\mu\nu}$ is also gauge invariant.

Now we can show that the energy density relation $T^{00} = \frac{1}{2}(E^2 + B^2)$ holds. The following relations hold:

$$F^{0i} = -E^i, \quad F^{0i} = F_0{}^i = -F_{0i} = -F^0{}_i, \quad F^{ij} = -F_i{}^j = F_{ij} = -\varepsilon_{ijk}B^k. \tag{4.20}$$

Than the zero component of the energy-momentum tensor is

$$\begin{aligned} \hat{T}^{00} &= F^{0i}\underbrace{F_i^0}_{F^{0i}} + \frac{1}{4}g^{00}(\underbrace{F_{0i}}_{-F^{0i}}F^{0i} + \underbrace{F_{i0}F^{i0}}_{-F^{i0}F^{i0}} + F_{ij}F^{ij}) \\ &= E^iE^i + \frac{1}{4}(-2E^iE^i + F^{ij}F^{ij}) \\ &= \frac{1}{2}E^iE^i + \frac{1}{4}\varepsilon^{ijk}B^k\varepsilon^{ijl}B^l = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2). \end{aligned} \tag{4.21}$$

Finally we can show

$$\hat{T}^{0i} = F_k{}^iF^{0k} = -E^k\varepsilon^{kij}B^j = \varepsilon^{ikj}E^kB^j = (\boldsymbol{E}\times\boldsymbol{B})^i = S^i. \tag{4.22}$$

Exercise 12: *Consider the action,*

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\bar{\lambda}}{4!} \phi^p \right), \tag{4.23}$$

with some power p for the nonlinear term. Study the behavior of the action under scale transformations (dilatations)

$$x \to \lambda x, \quad \phi(x) \to \lambda^{-D} \phi(\lambda^{-1} x), \quad \lambda > 0, \tag{4.24}$$

where D is a scaling exponent (dilatation weight) for the field ϕ. Under which conditions is the action scale invariant? Determine the corresponding conserved current using Noether's theorem.

Solution: Using the transformation of $\phi(x)$, the derivative transforms as

$$\partial_\mu \phi \to \lambda^{-D-1} \partial_\mu \phi. \tag{4.25}$$

Then, the transformation of the first part of the integral reads

$$\int d^4x \frac{1}{2} (\partial_\mu \phi(x))(\partial^\mu \phi(x)) \to \lambda^4 \int d^4x \, \lambda^{-2D-2} (\partial_\mu \phi(\lambda x))(\partial^\mu \phi(\lambda x)). \tag{4.26}$$

This part is invariant for a scaling exponent $D = 1$. Then the other parts of the integral become

$$\int d^4x \left(\frac{1}{2} m^2 \phi^2(x) + \frac{\bar{\lambda}}{4!} \phi^p(x) \right) \to \int d^4x \Bigg(\underbrace{\frac{1}{2} \lambda^{4-2} m^2 \phi^2}_{\text{invariant for } m=0} + \underbrace{\frac{\bar{\lambda}}{4!} \lambda^{4-p} \phi^p}_{\text{invariant for } p=4} \Bigg). \tag{4.27}$$

The invariant action can then be written as

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{\bar{\lambda}}{4!} \phi^4 \right). \tag{4.28}$$

For the Noether current we first need to calculate $\delta\phi$. For a small scale transformation $\lambda = 1 + \varepsilon$, the field transforms as

$$\begin{aligned} \phi \to \frac{1}{\lambda} \phi\left(\frac{x}{\lambda}\right) &= \frac{1}{1+\varepsilon} \phi(u) \quad \text{with} \quad u = \frac{x}{1+\varepsilon} \\ &= \frac{1}{1+\varepsilon} \phi(u) \bigg|_{\varepsilon=0} + \phi'(u) \bigg|_{\varepsilon=0} \varepsilon + \mathcal{O}\left(\varepsilon^2\right) \\ &= \phi(x) + \frac{-1}{(1+\varepsilon)^2} \phi\left(\frac{x}{1+\varepsilon}\right) \bigg|_{\varepsilon=0} \varepsilon \\ &= \phi(x) \underbrace{- \varepsilon \phi(x) - \varepsilon x^\nu \partial_\nu \phi(x)}_{\delta\phi}. \end{aligned} \tag{4.29}$$

Now, the change in the Lagrange density can be calculated as

$$\begin{aligned}
\delta\mathscr{L} &= \delta\left(\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\bar{\lambda}}{4!}\phi^4\right) = \left((\partial_\mu\phi)(\partial^\mu\delta\phi) - \frac{\bar{\lambda}}{3!}\phi^3\delta\phi\right) \\
&= -\varepsilon\left(2(\partial_\mu\phi)(\partial^\mu\phi) + (\partial_\mu\phi)(\partial_\mu\partial_\nu\phi) - \frac{\lambda}{3!}\phi^3x^\nu\partial_\nu\phi - 4\frac{\lambda}{4!}\phi^4\right) \\
&= -\varepsilon\left(4\mathscr{L} + x\partial_\nu\left(\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{\lambda}{4!}\phi^4\right)\right) = -\varepsilon(4\mathscr{L} + x^\nu\partial_\nu\mathscr{L}) \\
&= -\varepsilon\partial^\nu(x_\nu\mathscr{L}) = \partial_\nu K^\nu \quad \text{with} \quad K^\nu = -\varepsilon x^\nu\mathscr{L}.
\end{aligned} \tag{4.30}$$

Then the Noether current is determined by

$$J^\mu = \Pi^\mu\delta\phi - K^\mu \quad \text{with} \quad K^\mu = -\varepsilon x^\nu\mathscr{L} \quad \text{and} \quad \delta\phi = -\varepsilon(\phi(x) + x^\nu\partial_\nu\phi(x)). \tag{4.31}$$

5 Fifth Exercise sheet

Exercise 13: *Consider the Lagrangian for a triplet of real scalar fields ϕ^a, ($a = 1,2,3$), defining a classical field theory with rotational $O(3)$ invariance in field space,*

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - V(\phi^a \phi^a). \tag{5.1}$$

a) *Verify that the action is invariant under infinitesimal rotations in field space, which can be written analogously to rotations in coordinate space as $\phi^a \to \phi^a + \theta\varepsilon^{abc}\hat{n}^b\phi^c$, where $\hat{n}^b$ is a unit vector defining the rotation axis, and $\theta \ll 1$ is an infitesimal rotation angle.*

b) *Compute the Noether current and the Noether charge.*

c) *Verify the conservation of the Noether charge explicitly by using the equation of motion.*

Solution a): The first part of the Lagrangian transforms as follows:

$$\begin{aligned}(\partial_\mu \phi^a)(\partial^\mu \phi^a) \to &[(\partial_\mu \phi^a) + \theta\varepsilon^{abc}\hat{n}^b\partial_\mu\phi^c][(\partial^\mu \phi^a) + \theta\varepsilon^{abc}\hat{n}^b\partial^\mu\phi^c] \\ =&\big[(\partial_\mu \phi^a)(\partial^\mu \phi^a) + \underbrace{2\theta\varepsilon^{abc}\hat{n}^b(\partial_\mu \phi^a)(\partial^\mu \phi^c)}_{=0\ (a\leftrightarrow c)} + \mathcal{O}\big(\theta^2\big)\big].\end{aligned} \tag{5.2}$$

The expression $\phi^a\phi^a$ transforms as

$$\begin{aligned}\phi^a\phi^a \to &(\phi^a + \theta\varepsilon^{abc}\hat{n}^b\phi^a)(\phi^a + \theta\varepsilon^{abc}\hat{n}^b\phi^c) \\ =&\phi^a\phi^a + \underbrace{2\theta\varepsilon^{abc}\hat{n}^b\phi^c\phi^c}_{=0\ (a\leftrightarrow c)} + \mathcal{O}\big(\theta^2\big)\end{aligned} \tag{5.3}$$

$$\Rightarrow V(\phi^a\phi^a) \to V(\phi^a\phi^a). \tag{5.4}$$

Thus we find that $\mathscr{L}$ is invariant under infinitesimal rotations.

Solution b): From the lecture we can calculate the Noether current as

$$\begin{aligned}J^\mu &= \frac{\partial\mathscr{L}}{\partial\partial_\mu\phi^a}\delta\phi^a - \underbrace{K^\mu}_{=0} \\ &= \partial^\mu\phi^a\theta\varepsilon^{abc}\hat{n}^b\phi^c = \theta(\partial^\mu\phi^a)(\hat{n}\times\phi)^a.\end{aligned} \tag{5.5}$$

The surface term K^μ disappears for an invariant Lagrangian, since $\delta\mathscr{L} = \partial_\mu K^\mu$. In the following we drop θ by rewriting the Noether current $J^\mu = \theta j^\mu$. The Noether charge is then given by the integral of the zero-component of j^μ as

$$Q = \int \mathrm{d}^3x\, j^0 = \int \mathrm{d}^3x\,(\partial_0\phi^a)(\hat{n}\times\phi)^a = \int \mathrm{d}^3x\,\varepsilon^{abc}\dot{\phi}^a\hat{n}^b\phi^c. \tag{5.6}$$

Solution c): To determine the conservation of the Noether charge Q, we compute the equations of motion

$$
\begin{aligned}
\frac{\delta S}{\delta\phi^a} = 0 &\Leftrightarrow \frac{\partial\mathscr{L}}{\partial\phi^a} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi^a)}\right) = 0 \\
&\Leftrightarrow -\frac{\partial V}{\partial\phi^a} - \underbrace{\partial_\mu\partial^\mu}_{=\Box}\phi^a = 0 \\
&\Leftrightarrow -\frac{\partial V}{\partial(\phi^b\phi^b)}\frac{\partial(\phi^b\phi^b)}{\partial(\phi^a\phi^a)} + \Box\phi^a = 0.
\end{aligned} \tag{5.7}
$$

The time derivative of the Noether charge is

$$
\dot{Q} = \frac{\partial}{\partial t}\left[\int \mathrm{d}^3x\,\varepsilon^{abc}\hat{n}^b\dot{\phi}^a\phi^c\right] = \int \mathrm{d}^3x\,\varepsilon^{abc}\ddot{\phi}^a\phi^c\hat{n}^b + \int \mathrm{d}^3x\,\underbrace{\varepsilon^{abc}\hat{n}^b\dot{\phi}^a\dot{\phi}^c}_{=0\,(a\leftrightarrow c)}. \tag{5.8}
$$

The equations of motion can be decomposed as

$$
[\partial_0^2 - \partial_i^2 + V']\phi^a = 0 \Leftrightarrow \ddot{\phi}^a = (\partial_i^2\phi^a) - V'\phi^a. \tag{5.9}
$$

Inserting them into $\dot{Q}$ leads to

$$
\begin{aligned}
\dot{Q} &= \int_V \mathrm{d}^3x\,\varepsilon^{abc}\partial_i(\partial_i\phi^a)\phi^c\hat{n}^b - \int_V \mathrm{d}^3x\,\underbrace{\varepsilon^{abc}\hat{n}^bV\phi^a\phi^b}_{=0\,(a\leftrightarrow c)} \\
&\overset{\text{P.I.}}{=} \varepsilon^{abc}\hat{n}^b\underbrace{(\partial_i\phi^a)\phi^c}_{=0}\Big|_{\partial V} - \int \mathrm{d}^3x\,\varepsilon^{abc}\underbrace{(\partial_i\phi^a)(\partial_i\phi^c)}_{=0\,(a\leftrightarrow c)}\hat{n}^b = 0.
\end{aligned} \tag{5.10}
$$

Exercise 14: *Consider an almost $O(N)$ invariant scalar model with Lagrangian*

$$
\mathscr{L} = \frac{1}{2}(\partial_\mu\phi^a)(\partial^\mu\phi^a) - \underbrace{\left(-\frac{1}{2}\mu^2\phi^a\phi^a + \frac{\lambda}{4!}(\phi^a\phi^a)^2\right)}_{V(\phi^a\phi^a)} - \delta V, \tag{5.11}
$$

where $a = 1,\ldots,N$ and δV is a potential term that breaks $O(N)$ symmetry explicitly. Upon spanning ϕ^a by a parametrization $\phi = (\pi^i, \Sigma)$, where $i = 1,\ldots,N-1$, δV takes the form $\delta V = -h\Sigma$ with a positive constant parameter $h > 0$.

a) *Determine the position of the global minimum of the potential to first order in h.*

b) *Verify that the would-be Goldstone bosons acquire a mass. Compute the mass to first order in h.*

Solution a): We start by writing the potential with the parametrization $\phi = (\pi^i, \Sigma)$ as

$$
V = -\frac{1}{2}\mu^2\pi^2 - \frac{1}{2}\mu^2\Sigma^2 + \frac{\lambda}{4!}(\pi^2 + \Sigma^2)^2 - h\Sigma. \tag{5.12}
$$

Schematically, the potential term δV tilts the potential towards the positive Σ axis. Thus, the global minimum is at $\pi^i = 0$ and at a finite Σ_0. Hence, the global minimum corresponds to

$$0 = \left.\frac{\partial V}{\partial \Sigma}\right|_{\pi^i=0,\Sigma_0} \Leftrightarrow \left(\frac{\lambda}{3!}\Sigma_0^2 - \mu^2\right)\Sigma_0 - h = 0. \tag{5.13}$$

In case of $h = 0$ this yields

$$\Sigma_0^{h=0} = \sqrt{\frac{6\mu^2}{\lambda}}. \tag{5.14}$$

Now we make the ansatz that for nonzero h the solution Σ_0 can be written as

$$\Sigma_0 = \Sigma_0^{h=0} + \alpha h. \tag{5.15}$$

Inserting this into the above relation (5.13) yields

$$0 = \left[-\mu^2 + \frac{\lambda}{3!}\left(\frac{6\mu^2}{\lambda} + 2\sqrt{\frac{6\mu^2}{\lambda}}\alpha h + \mathcal{O}\big(h^2\big)\right)\right] \cdot \left[\sqrt{\frac{6\mu^2}{\lambda}} + \mathcal{O}(h)\right] - h. \tag{5.16}$$

This can be solved for α

$$\alpha = \frac{1}{2\mu^2} \Leftrightarrow \Sigma_0 = \sqrt{\frac{6\mu^2}{\lambda}} + \frac{h}{2\mu^2}, \tag{5.17}$$

which is now a global minimum to $\mathcal{O}(h)$ order.

Solution b): For this discussion we again define $\Sigma = \Sigma_0 + \sigma(x)$. The potential V can then be written as

$$V = -\frac{1}{2}\mu^2\pi^2 - \frac{1}{2}\mu^2(\Sigma_0 + \sigma(x))^2 + \frac{\lambda}{4!}(\pi^2 + (\Sigma_0 + \sigma)^2)^2 - h(\Sigma_0 + \sigma(x)). \tag{5.18}$$

Now, the pseudo-Goldstone boson aquires a mass term m_π^2 which is quadratic in the π^i field. Neglecting all other terms leads to

$$\begin{aligned}
V &= -\frac{1}{2}\mu^2\pi^2 + \frac{2\lambda}{4!}\pi^2\Sigma_0^2 + \mathcal{O}\big(\pi^4, \pi^4\sigma, \pi^2\sigma\big)\\
&\qquad \text{Calculate:}\quad \Sigma_0^2 = \left(\sqrt{\frac{6\mu^2}{\lambda}} + \frac{h}{2\mu^2}\right)^2 = \frac{6\mu^2}{\lambda} + \sqrt{\frac{6\mu^2}{\lambda}}\frac{h}{\mu^2} + \mathcal{O}\big(h^2\big)\\
&= \underbrace{-\frac{1}{2}\mu^2\pi^2 + \frac{2\lambda}{4!}\frac{6\mu^2}{\lambda}\pi^2}_{=0} + \frac{2\lambda}{4!}\sqrt{\frac{6\mu^2}{\lambda}}\frac{h}{\mu^2}\pi^2 + \mathcal{O}\big(h^2, \pi^4, \pi^4\sigma, \pi^2\sigma\big)\\
&= \frac{1}{2}h\sqrt{\frac{\lambda}{6\mu^2}}\pi^2 = \frac{1}{2}m_\pi^2\pi^2 \quad \text{with} \quad m_\pi^2 = h\sqrt{\frac{\lambda}{6\mu^2}}.
\end{aligned} \tag{5.19}$$

Exercise 15: *Consider the action for a free $O(N)$ symmetric field theory,*

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a). \tag{5.20}$$

a) *Convince yourself that the theory becomes interacting by simply imposing the constraint $\phi^a\phi^a = 1$. For this, use the parametrization of exercise 14, eliminate the Σ field by the constraint, and compute the leading interactions for small π^i fluctuations.*

b) *This model is called a nonlinear σ model. Construct a suitable limit procedure such that the nonlinear model arises from the linear σ model (of exercise 14 with $\delta V = 0$).*

Solution a): With the constraint $\phi^a\phi^a = 1$ and $\phi = (\pi^i, \Sigma)$, the Σ field can be written as

$$\Sigma^2 + (\pi^i)^2 = 1 \quad \Rightarrow \quad \Sigma = \sqrt{1-(\pi^i)^2} \tag{5.21}$$

$$\Rightarrow \partial_\mu \Sigma = -\frac{1}{\sqrt{1-(\pi^i)^2}}(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi}). \tag{5.22}$$

Then the Lagrangian transforms as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi}) + \frac{1}{2}(\partial_\mu\Sigma)(\partial^\mu\Sigma)\\ &= \frac{1}{2}(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi}) + \frac{1}{2}\frac{(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2}{1-\pi^2}\\ &= \frac{1}{2}(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi}) + \frac{1}{2}(\boldsymbol{\pi}\partial_\mu\boldsymbol{\pi})^2 + \dots.\end{aligned} \tag{5.23}$$

Solution b): We start with the Lagrangian with a potential (with $\delta V = 0$) previously given by (5.12)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\boldsymbol{\pi})(\partial^\mu\boldsymbol{\pi}) + \frac{1}{2}(\partial_\mu\Sigma)(\partial^\mu\Sigma) + \underbrace{\frac{1}{2}\mu^2\pi^2 + \frac{1}{2}\mu^2\Sigma^2 - \frac{\lambda}{4!}(\pi^2+\Sigma^2)^2}_{-V}. \tag{5.24}$$

Again we demand the following

$$\left.\frac{\partial V}{\partial \Sigma}\right|_{\pi^i=0,\Sigma_0} = 0 \quad \Rightarrow \quad \Sigma_0 = \sqrt{\frac{6\mu^2}{\lambda}}. \tag{5.25}$$

Then we find with the given constraint $\phi^a\phi^a = 1$

$$\left.\phi^a\phi^a\right|_{\text{min}} = 1 = \left.\Sigma^2 + \pi^2\right|_{\text{min}} = \Sigma_0^2 \quad \Rightarrow \quad \lambda = 6\mu^2. \tag{5.26}$$

Now defining $\Sigma = v + \sigma(x)$ we have

$$\begin{aligned}V &= -\frac{1}{2}\mu^2\pi^2 - \mu^2 v\sigma - \frac{1}{2}\mu^2\sigma^2 + \frac{\lambda}{4!}\pi^4 + \frac{\lambda}{4!}\left(4v\sigma^3 + 6v^2\sigma^2 + 4v^3\sigma + \sigma^4\right) + \frac{2\lambda}{4!}\pi^2(v + 2v\sigma + \sigma^2)\\ &= \underbrace{\frac{1}{2}(-\mu^2 + \frac{\lambda}{2}v^2)\sigma^2}_{\text{mass of sigma}} + \underbrace{\frac{\lambda}{4!}\pi^4 + \frac{\lambda}{4!}\sigma^4 + \frac{\lambda}{6}v\sigma^3 + \frac{\lambda}{6}v\pi^2\sigma + \frac{\lambda}{12}\pi^2\sigma^2}_{\text{interaction}}.\end{aligned} \tag{5.27}$$

Then, using (5.26), the sigma mass can be determined to be

$$m_\sigma^2 = -\mu^2 + \frac{\lambda}{2} v^2 = 2\mu^2 \quad \text{with} \quad v = \frac{6\mu^2}{\lambda} = 1. \tag{5.28}$$

With the constraint $\pi^2 + \Sigma^2 = 1$ and for $\pi^2 \ll 1$, the $\sigma(x)$ must be very small $\sigma(x) \lll 1$. For a small width σ, the curvature of $V(\Sigma)$ has to go to infinity

$$V''(\Sigma)\bigg|_{\Sigma=v} = m_\sigma^2 \to \infty. \tag{5.29}$$

6 Sixth Exercise sheet

Exercise 16: *The nonrelativistic version of scalar QED,*

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \psi^* \mathrm{i}\partial_t\psi - \frac{1}{2m}(\boldsymbol{D}\psi)^* \cdot \boldsymbol{D}\psi - V(\psi^*\psi), \tag{6.1}$$

where $\boldsymbol{D} = \nabla - \mathrm{i}q\boldsymbol{A}$, can be used to describe superconductors. Here, the complex scalar field ψ should be thought of as the wave function of the coherent bosonic state that describes the Cooper pairs; hence, its charge is $q = 2e$. The model has a local gauge invariance.

a) *Local gauge invariance also implies a global phase invariance of the scalar field, $\psi \to \mathrm{e}^{-\mathrm{i}\alpha}\psi$. Compute the Noether current J^μ of this symmetry. Verify that the spatial components (up to the infinitesimal symmetry parameter α) agree with the Cooper current*

$$\boldsymbol{j} = \frac{\mathrm{i}}{2m}((\boldsymbol{D}\psi)^*\psi - \psi^*\boldsymbol{D}\psi). \tag{6.2}$$

b) *Assume that the potential V inside a superconductor has a minimum at a finite field amplitude at $|\psi| > 0$. Verify that the generic form of the wave function in this ground state then is $\psi = \sqrt{|\rho|}\mathrm{e}^{\mathrm{i}\varphi}$, where ρ agrees with the Noether current density up to the infinitesimal parameter and $\varphi = \varphi(\boldsymbol{x}, t)$ is an arbitrary phase.*

c) *Compute the Cooper current for this ground state of a constant density ρ = const. and verify that the maxwell equation $\vec{\boldsymbol{\nabla}} \times \boldsymbol{B} = \boldsymbol{j}$ implies the London equation*

$$\boldsymbol{\nabla}^2 = \frac{1}{\lambda_L}\boldsymbol{B}. \tag{6.3}$$

Compute λ_L^2 and convince yourself that λ_L can be interpreted as the penetration depth of a magnetic field into a superconductor, thus explaining the Meißner-Ochsenfeld effect. How is the penetration depth related to the photon mass?

Solution a): The global phase variance for a small angle α can be written as

$$\psi \to \mathrm{e}^{-\mathrm{i}\alpha}\psi = \psi - \mathrm{i}\alpha\psi \Rightarrow \delta\psi = -\mathrm{i}\alpha\psi \tag{6.4}$$

$$\psi^* \to \mathrm{e}^{\mathrm{i}\alpha}\psi* = \psi^* + \mathrm{i}\alpha\psi^* \Rightarrow \delta\psi^* = \mathrm{i}\alpha\psi^*. \tag{6.5}$$

According to the lecture, the Noether current $J^\mu = (\rho, \boldsymbol{J})$ is given by

$$J^\mu = \frac{\partial\mathscr{L}}{\partial(\partial^\mu\psi)}\delta\psi + \frac{\partial\mathscr{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^*. \tag{6.6}$$

The zero component can be calculated as

$$\rho = \underbrace{\frac{\partial\mathscr{L}}{\partial(\partial_t\psi)}}_{=\mathrm{i}\psi^*} \underbrace{\delta\psi}_{=-\mathrm{i}\alpha\psi} + \underbrace{\frac{\partial\mathscr{L}}{\partial(\partial_t\psi^*)}}_{=0}\delta\psi^* = \alpha\psi^*\psi. \tag{6.7}$$

The other components are computed as

$$J^i = \underbrace{\frac{\partial \mathscr{L}}{\partial(\partial_i \psi)}} \delta\psi + \frac{\partial \mathscr{L}}{\partial(\partial_i \psi^*)}\delta\psi^*$$

$$= -\frac{1}{2m}(D^j\psi)^* \frac{\partial(D^j\psi)}{\partial(\partial_i\psi)} = -\frac{1}{2m}(D^j\psi)^* \underbrace{\frac{\partial}{\partial(\partial_i\psi)}[\partial^j\psi - \mathrm{i}qA^j]}_{=1} = -\frac{1}{2m}(D^j\psi)^*$$

$$= \frac{\mathrm{i}\alpha}{2m}[(D^j\psi)^*\psi - \psi^*(D^j\psi)]. \tag{6.8}$$

Solution b): Assuming that $V(\psi^*\psi)$ has a minimum at $\psi_0 : |\psi_0| > 0$, we find

$$|\psi_0|^2 = \rho > 0. \tag{6.9}$$

Then, for the ground state, we require $|\psi_0(\boldsymbol{x}, t)| = \sqrt{\rho}$. Due to local gauge invariance $\psi \to \mathrm{e}^{-\mathrm{i}\alpha(\boldsymbol{x},t)}\psi$, then we have the freedom to add a phase term

$$\psi_0(\boldsymbol{x}, t) = \sqrt{\rho}\mathrm{e}^{\mathrm{i}\phi(\boldsymbol{x},t)}. \tag{6.10}$$

Solution c): The Cooper current for the minimum $\psi_0(\boldsymbol{x}, t)$ for $\rho = \text{const.}$ reads

$$\begin{aligned}\boldsymbol{J} &= \frac{\mathrm{i}}{2m}\left[\vec{\nabla}\left(\sqrt{\rho}\mathrm{e}^{-\mathrm{i}\phi}\right)\sqrt{\rho}\mathrm{e}^{\mathrm{i}\phi} - \sqrt{\rho}\mathrm{e}^{-\mathrm{i}\phi}(\vec{\nabla}\sqrt{\rho}\mathrm{e}^{\mathrm{i}\phi}) + \mathrm{i}q\boldsymbol{A}\psi^*\psi + \mathrm{i}q\boldsymbol{A}\psi^*\psi\right] \\ &= \frac{\rho}{m}[\vec{\nabla}\phi - q\boldsymbol{A}].\end{aligned} \tag{6.11}$$

From Maxwell's equations $(\vec{\nabla}\cdot\boldsymbol{B}) = 0$ and $(\vec{\nabla}\times\boldsymbol{B}) = \boldsymbol{J}$ we find

$$\vec{\nabla}\times\left(\vec{\nabla}\times\boldsymbol{B}\right) = \vec{\nabla}\times\boldsymbol{J} = \frac{\rho}{m}\Big[\underbrace{\vec{\nabla}\times\left(\vec{\nabla}\phi\right)}_{=0} - q\underbrace{(\vec{\nabla}\times\boldsymbol{A})}_{=\boldsymbol{B}}\Big] = -\frac{q\rho}{m}\boldsymbol{B} \tag{6.12}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\boldsymbol{B}\right) = \underbrace{\vec{\nabla}\left(\vec{\nabla}\cdot\boldsymbol{B}\right)}_{=0} - \Delta\boldsymbol{B}. \tag{6.13}$$

Combining both yields then

$$\Delta\boldsymbol{B} = \frac{\rho q}{m}\boldsymbol{B} \quad \text{London equation,} \tag{6.14}$$

where $\lambda_L = \sqrt{\frac{m}{q\rho}}$ is the London penetration length. In order to justify the interpretation of λ_L, we consider the boundary of a super conductor. Outside the super conductor $\boldsymbol{B}$ propagates in an arbitrary direction $|\boldsymbol{B}| = |\boldsymbol{B}_0|$. In the z-direction of the super conductor $\boldsymbol{B}$ should satisfy

$$\frac{\partial^2}{\partial z^2}\boldsymbol{B} = \frac{1}{\lambda_L^2}\boldsymbol{B} \quad \Rightarrow \quad \boldsymbol{B} = \boldsymbol{B}_0\mathrm{e}^{-\frac{z}{\lambda_L}}. \tag{6.15}$$

From here we can see that λ_L describes the depth of penetration of the magnetic field inside the super conductor. The magnetic field depletes quickly inside the super conductor. The conservation of magnetic flux in combination with the London equation leads to the *Meißner-Ochsenfeld effect*

$$\phi_B = \iint_{\text{Surf.}} \boldsymbol{B}\,\mathrm{d}s \Rightarrow \phi_B = 0. \tag{6.16}$$

The magnetic field lines are ejected from the super conductor. The conclusion holds true since the magnetic field can only penetrate the super conductor until the penetration length but the same number of magnetic field liens that enter the super conductor leaves as well.

Finally, we relate λ_L with the photon mass

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{2m}(\boldsymbol{D}\psi)^*(\boldsymbol{D}\psi) + \ldots \\
&= -\frac{1}{2m} q^2 \rho \boldsymbol{A}^2 + \ldots \\
&= -\frac{1}{2m} m_A^2 \boldsymbol{A}^2 \quad \Rightarrow \quad m_A^2 = \frac{q^2\rho}{m} = \frac{q}{\lambda_L}.
\end{aligned} \tag{6.17}$$

The photon mass corresponds to the inverse penetration length. This means the Meißner effect can be explained by assuming that photons inside a super conductor acquire an effective mass. This is linked to the modified Maxwell's equations, i. e. Proca theory.

Exercise 17: *Consider an $O(N)$ invariant scalar model with Lagrangian*

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - V(\rho), \quad \rho = \frac{1}{2}\phi^a\phi^a, \tag{6.18}$$

where $a = 1, \ldots, N$. Given the potential with some minimum ρ_0, any concrete parametrization of ρ_0 in field space is legitimate and physically equivalent. Thus, the eigenvalues of the mass matrix $m_{ab}^2 = \partial^2 V / \partial\phi^a \partial\phi^b$ can be written in terms of $O(N)$-invariant quantities. Diagonalize the mass matrix in terms of such invariant expressions. The final expression should also hold for the case $\rho_0 = 0$.

Solution: Let us rewrite the masss matrix as follows

$$m_{ab}^2 = \frac{\partial^2 V(\rho)}{\partial\phi^a\partial\phi^b} = \frac{\partial}{\partial\phi^b}\left(\frac{\partial\rho}{\partial\phi^a}\frac{\partial V}{\partial\rho}\right) = \frac{\partial^2\rho}{\partial\phi^b\partial\phi^a}\frac{\mathrm{d}V}{\mathrm{d}\rho} + \frac{\partial\rho}{\partial\phi^a}\frac{\partial\rho}{\partial\phi^b}\frac{\mathrm{d}^2 V}{\mathrm{d}\rho^2}. \tag{6.19}$$

Let us calculate the different derivatives separately:

$$\frac{\partial\rho}{\partial\phi^a} = \frac{\partial(\frac{1}{2}\phi^b\phi^b)}{\partial\phi^a} = \frac{1}{2}\phi^b\delta^{ab} + \frac{1}{2}\delta^{ab}\phi^b = \phi^a \tag{6.20}$$

$$\frac{\partial^2\rho}{\partial\phi^a\partial\phi^b} = \frac{\partial}{\partial\phi^b}(\phi^a) = \delta^{ab}. \tag{6.21}$$

The the mass matrix can be written as

$$m^2_{ab} = \frac{\mathrm{d}V}{\mathrm{d}\rho}\delta^{ab} + \phi^a\phi^b\frac{\mathrm{d}^2V}{\mathrm{d}\rho^2}. \tag{6.22}$$

m^2_{ab} is the matrix we wish to diagonalize and determine its eigenvalues/eigenvectors. Let us compute

$$\begin{aligned} m^2_{ab}\phi^b &= \frac{\mathrm{d}^2V}{\mathrm{d}\rho^2}\phi^a\underbrace{\phi^b\phi^b}_{2\rho} + \frac{\mathrm{d}V}{\mathrm{d}\rho}\phi^a \\ &= [2\rho V''(\rho) + V'(\rho)]\phi^a. \end{aligned} \tag{6.23}$$

hence, ϕ^b is an eigenvector of m^2_{ab} with eigenvalues $\lambda = 2\rho V''(\rho) + V'(\rho)$.

Let us now introduce a new field parallel to the field $\boldsymbol{\phi}$ as

$$\begin{aligned} (\boldsymbol{\chi}_\parallel)^a &= \chi_\parallel\frac{\phi^a}{|\boldsymbol{\phi}|} \quad \text{with} \quad \chi_\parallel = \frac{\chi^b\phi^b}{|\boldsymbol{\phi}|} \\ &= \frac{\phi^a}{|\boldsymbol{\phi}|}\frac{\phi^a}{|\boldsymbol{\phi}|} = \frac{\frac{1}{2}\phi^a\phi^b}{\frac{1}{2}|\boldsymbol{\phi}|^2}\chi^b \\ &= \frac{1}{2}\phi^a\phi^b\frac{1}{\rho}\chi^b = \rho^{ab}_\parallel\chi^b. \end{aligned} \tag{6.24}$$

For a generic field vector $\boldsymbol{\chi}$, the perpendicular part can be extracted via

$$\boldsymbol{\chi}_\perp = \boldsymbol{\chi} - \boldsymbol{\chi}_\parallel = \mathbb{1}\boldsymbol{\chi} - \rho_\parallel\boldsymbol{\chi} = (1-\rho_\parallel)\boldsymbol{\chi} = \rho_\perp\boldsymbol{\chi} \quad \Rightarrow \quad \rho^{ab}_\perp = \delta^{ab} - \frac{1}{2}\frac{\phi^a\phi^b}{\rho}. \tag{6.25}$$

The $\rho_\perp$ and $\rho_\parallel$ matrices fulfill the following relations:

$$(\rho^2_\parallel)^{ab} = \rho^{ab}_\parallel\rho^{bc}_\parallel = \frac{1}{4}\frac{\phi^a\phi^b\phi^b\phi^c}{\rho^2} = \frac{1}{2}\frac{\phi^a\rho\phi^c}{\rho^2} = \rho^{ac}_\parallel \tag{6.26}$$

$$(\rho^2_\perp)^{ab} = \ldots = \rho^{ab}_\perp. \tag{6.27}$$

Furthermore, they have the following properties:

$$\rho^{ab}_\parallel\rho^{bc}_\perp = \frac{1}{2}\frac{\phi^a\phi^b}{\rho}\left(\delta^{bc} - \frac{\phi^b\phi^c}{2\rho}\right) = \frac{1}{2}\frac{\phi^a\phi^c}{\rho} - \frac{1}{2}\frac{\phi^a\phi^c}{\rho} = 0 \tag{6.28}$$

$$\rho^{ab}_\parallel + \rho^{bc}_\perp = \delta^{ab}. \tag{6.29}$$

We can do this to construct a diagonalized matrix as

$$\hat{A} = \alpha\rho_\parallel + \beta\rho_\perp = \begin{pmatrix} \alpha & 0 & \ldots & 0 \\ 0 & \beta & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \beta \end{pmatrix}. \tag{6.30}$$

Then we have

$$\hat{A}\boldsymbol{\chi}_{\parallel} = (\alpha\rho_{\parallel} + \beta\rho_{\perp})(\rho_{\parallel}\boldsymbol{\chi}) = \alpha\rho_{\parallel}\boldsymbol{\chi} = \alpha\boldsymbol{\chi}_{\parallel} \quad \text{with} \quad \boldsymbol{\chi}_{\parallel} = \begin{pmatrix}\rho\\0\\\vdots\\0\end{pmatrix} \tag{6.31}$$

$$\hat{A}\boldsymbol{\chi}_{\perp} = (\alpha\rho_{\parallel} + \beta\rho_{\perp})(\rho_{\perp}\boldsymbol{\chi}) = \beta\rho_{\perp}\boldsymbol{\chi} = \beta\boldsymbol{\chi}_{\perp} \quad \text{with} \quad \boldsymbol{\chi}_{\perp} = \begin{pmatrix}0\\\pi_1\\\vdots\\\pi_{N-1}\end{pmatrix}. \tag{6.32}$$

Then we can write equation (6.22) as

$$\begin{aligned} m^2_{ab} &= V''(\rho)\phi^a\phi^b + V'(\rho)\delta^{ab} \\ &= V''(\rho)2\rho\rho_{\parallel}^{ab} + V'(\rho)(\rho_{\parallel}^{ab} + \rho_{\perp}^{ab}) &\quad (6.33) \\ &= [V''(\rho)2\rho + V'(\rho)]\rho_{\parallel}^{ab} + V'(\rho)\rho_{\perp}^{ab}. &\quad (6.34) \end{aligned}$$

Now, we specify the potential as

$$V(\rho) = \frac{1}{2}\gamma\phi^a\phi^a + \frac{\lambda}{4!}(\phi^a\phi^a)^2 = \gamma\rho + \frac{\lambda}{3!}\rho^2. \tag{6.35}$$

The derivatives yield

$$V'(\rho) = \gamma + \frac{\lambda}{3}\rho, \qquad V''(\rho) = \frac{\lambda}{3}. \tag{6.36}$$

We can consider two different cases:

1. Case: $\gamma = m^2 > 0$ (symmetric)

$$\frac{\partial V(\rho)}{\partial\phi^a} = \phi^a\frac{\partial V(\rho)}{\partial\rho} \overset{!}{=} 0 \Rightarrow \phi^a = 0 \Rightarrow \rho = 0. \tag{6.37}$$

   Then the mass of the σ and π bosons is given as

$$m^2_\sigma\Big|_{\rho=0} = (V''(0)2\cdot 0 + V'(0)) = \gamma = m^2, \tag{6.38}$$

$$m^2_\pi\Big|_{\rho=0} = V'(0) = \gamma = m^2. \tag{6.39}$$

2. Case: $\gamma = -\mu^2 < 0$ (broken)

$$\frac{\partial V(\rho)}{\partial\phi^a} = \phi^a\frac{\partial V(\rho)}{\partial\rho} \overset{!}{=} 0 \Rightarrow \rho = \frac{3\mu^2}{\lambda}. \tag{6.40}$$

   Then we can again calculate the mass of the σ and π bosons:

$$m^2_\sigma\Big|_{\rho=\rho_0} = 2\mu^2, \tag{6.41}$$

$$m^2_\pi\Big|_{\rho=\rho_0} = V'(\rho_0) = 0. \tag{6.42}$$

   We find $(N-1)$ massless Goldstone bosons. This agrees with the results in the lecture.

Exercise 18: *Consider the action for a complex two-component scalar field $\phi_i \in \mathbb{C}$, $i = 1,2$, which may be summarized in a complex vector $\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$,*

$$\mathscr{L} = (\partial_\mu \phi_i^*)(\partial^\mu \phi_i) + \mu^2 \phi_i^* \phi_i - \frac{\lambda}{3!}(|\phi_1|^2 + |\phi_2|^2)^2. \tag{6.43}$$

- ***a)*** *Identify the global symmetry group of rotations in the complex field space.*
- ***b)*** *For $\mu^2 > 0$, the potential has minima at nonzero field values. Determine this submanifold in field space.*
- ***c)*** *Select a possible vacuum state Φ_0. What is the symmetry group that leaves this ground state invariant?*
- ***d)*** *Now expand the field in terms of real fields denoting excitations on top of the vacuum. Determine the number of Goldstone modes and compare this number to the number of "broken generators".*

Solution a): Let us replace $\phi_i \in \mathbb{C}$ by real fields $\phi^a \in \mathbb{R}$ with $i = 1,2,3,4$ as

$$\begin{aligned}\phi_1 &= \frac{1}{\sqrt{2}}(\phi^1 + \mathrm{i}\phi^2), \quad \phi_1^* = \frac{1}{\sqrt{2}}(\phi^1 - \mathrm{i}\phi^2)\\ \phi_2 &= \frac{1}{\sqrt{2}}(\phi^3 + \mathrm{i}\phi^4), \quad \phi_2^* = \frac{1}{\sqrt{2}}(\phi^3 - \mathrm{i}\phi^4).\end{aligned} \tag{6.44}$$

Then, we wish to rewrite $\mathscr{L}$ in terms of $\phi^a \in \mathbb{R}$. Note that

$$\phi_i^* \phi_i = |\phi_1|^2 + |\phi_2|^2 = \frac{1}{2}[(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 + (\phi^4)^2] = \frac{1}{2}\phi^a \phi^a. \tag{6.45}$$

Now, $\mathscr{L}$ can be written as

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) + \frac{1}{2}\mu^2 \phi^a \phi^a - \frac{\lambda}{4!}(\phi^a \phi^a)^2. \tag{6.46}$$

This corresponds to an $O(4)$-invariant model.

Solution b): For $\mu^2 > 0$, the global minimum can be determined as

$$\begin{aligned}\frac{\partial V(\phi^a\phi^a)}{\partial \phi^b} &= \mu^2 \phi^b - \frac{\lambda}{4!} 2(\phi^a\phi^a)(2\phi^b)\\ &= \left[\mu^2 - \frac{\lambda}{3!}(\phi^a\phi^a)\right]\phi^b\\ &\overset{\min}{=} \phi_0^a \phi_0^a = (\phi_0^1)^2 + (\phi_0^2)^2 + (\phi_0^3)^2 + (\phi_0^4)^2 = \frac{6\mu^2}{\lambda} := v.\end{aligned} \tag{6.47}$$

This corresponds to the 3-dimensional boundary of a 4-ball also known as a 3-sphere.

Solution c): We choose the vacuum $\phi_0^a = \begin{pmatrix} 0 & 0 & 0 & v \end{pmatrix}^T$. ϕ_0^a is left invariant by all the 4×4 matrices of the form

$$\begin{pmatrix} & & & 0 \\ & \theta & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } \theta : 3 \times 3 \text{ orthogonal matrices, i. e. } \theta^T \theta = \mathbb{1}. \tag{6.48}$$

Thus, there is a residual $O(3)$ symmetry.

Solution d): We do the same as in the lecture with $\phi^a = \begin{pmatrix} \pi \\ v \end{pmatrix}$, then by rewriting the Lagrangian we obtain:

- 3 Goldstone bosons (massless) π^i, $i = 1, 2, 3$, due to the spontaneous symmetry breaking
- Massive radial mode $m_\sigma^2 = 2\mu^2$.

The linearly independent $O(4)$ rotations that do not leave ϕ_0^a invariant correspond to those where the 4th-component is the same with the first ones i. e.

$$U_1 = \begin{pmatrix} \cos\phi & 0 & 0 & -\sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sin\phi & 0 & 0 & \cos\phi \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 & 00 & \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$U_3 = \begin{pmatrix} 1 & 0 & 00 & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & 0 & \cos\phi \end{pmatrix} \tag{6.49}$$

The three broken symmetry transformation correspond to the number of Goldstone bosons. We can now introduce the generators of $O(4)$ rotations

$$K_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{6.50}$$

which generate the broken $O(4)$ transformations $U_i = \mathrm{e}^{\alpha K_i}$. Then,

$$\phi^a \to (U_i \phi)^a \overset{\alpha \ll 1}{\approx} \phi^a + \alpha K_i^{ab} \phi^b \Leftrightarrow \delta\phi^a = \alpha K_i^{ab} \phi^b \tag{6.51}$$

with K_i being the broken generators.