Friedrich-Schiller-Universität Jena
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Laser Engineering

All Exercises

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Contents

1 Problem sheet 1

1.1 Exercise 1 - Lambert Beer's law

YB:YAG has an absorption cross section of $0{,}8 \cdot 10^{-20}$ cm^2 at 940 nm. A crystal manufacturer offers you crystals with a doping concentration of 3 atomic percent ($4{,}11 \cdot 10^{20}\,\frac{1}{\text{cm}^3}$).

a) *You want to absorb 90 % of your pump radiation. Assuming Lambert Beer's law, how thick must be the crystal? (neglect any changes in the density of excited ions).*

b) *The effective emission cross section of Yb:YAG is $2 \cdot 10^{20}$ cm^2 at 1030 nm. How much is the small signal gain for a single pass through this crystal if 20 % of the ions are in the excited state?*

c) *Assuming negligible absorption at the laser wavelength, how big is the saturation fluence at laser wavelength? What is the meaning of this parameter for the laser design?*

d) *How big is the heat fraction of this laser? How much power is transferred into heat for 1 W of incident power. Assume that all emitted radiation is transferred to 1030 nm.*

a.) Solution:

Lambert Beer's law states that the intensity of the pump radiation decreases exponentially with the absorption coefficient $\alpha = N_1 \cdot \sigma_a$

$$I_p = I_0 \mathrm{e}^{-\alpha z}, \quad \text{with} \quad \alpha = 4{,}11 \cdot 10^{20}\,\frac{1}{\text{cm}^3} 0{,}8 \cdot 10^{-20}\,\text{cm}^2 = 3{,}22\,\frac{1}{\text{cm}}. \tag{1.1}$$

We now demand that the pump intensity is reduced to 10 % of its initial value and solve for z

$$\begin{aligned} I_p \stackrel{!}{=} 0.1 I_0 \quad \Rightarrow \quad & \ln\left(\frac{1}{10}\right) = -\alpha z \\ & \Rightarrow z = \frac{\ln 10}{\alpha} = 0{,}72\,\text{cm}. \end{aligned} \tag{1.2}$$

b.) Solution:

The small signal gain is defined as $g = N_2 \sigma_e$, where N_2 is the number density of ions in the excited state

$$g = N_2 \sigma_e = 0.2 \cdot N_1 \sigma_e = 0.2 \cdot 4{,}11 \cdot 10^{20}\,\frac{1}{\text{cm}^3} \cdot 2 \cdot 10^{-20}\,\text{cm}^2 = 1{,}644\,\frac{1}{\text{cm}}. \tag{1.3}$$

c.) Solution:

At the laser wavelength the absorption cross section is assumed to be zero. The saturation fluence is then given as

$$F_{\text{sat}} = \frac{h\nu}{\sigma_a + \sigma_e} = \frac{1{,}204\,\text{eV}}{2 \cdot 10^{-20}\,\text{cm}^2} = 9{,}645\,\frac{\text{J}}{\text{cm}}. \tag{1.4}$$

The saturation fluence describes the fluence needed to reduce the laser medium gain by a factor of $1/e$ within a single pass. The laser should operate with an extraction fluence $F_{\text{extract}} \geq F_{\text{Sat}}$.

d.) Solution:

The heat fraction is given as the energy lost between pumping and the laser transition due to nonradiative process from the pump level to the upper laser level. It is given as the following ratio

$$\eta = \frac{h(\nu_p - \nu_l)}{h\nu_p} = 1 - \frac{\nu_l}{\nu_p} = 1 - \frac{\lambda_p}{\lambda_l} = 1 - \frac{940\,\text{nm}}{1030\,\text{nm}} = 8{,}7\,\%. \tag{1.5}$$

The amount of power transferred to heat is then

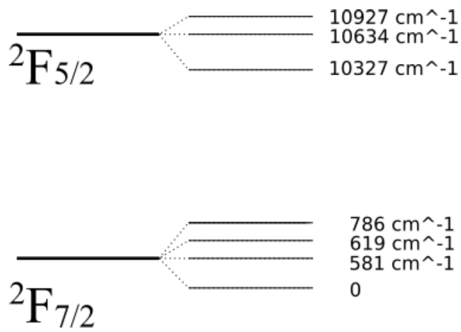
$$Q = \eta \cdot P_{\text{in}} = 0{,}087\,\text{W}. \tag{1.6}$$

2 Problem sheet 2

2.1 Exercise 2 - Cross sections

Imagine you want to investigate the spectroscopic properties of an Yb:YAG sample with an experimental setup. You get three spectra: "reference.csv" which shows the spectrum from the white ligth source without any sample, "absorption.csv" that shows the same light source after passing the crystal and being partially absorbed by it and "fluorescence.csv" which shows the measured fluorescence spectrum of the sample when stimulated with the fiber coupled laser diode. The task is now to calculate the absorption and emission cross sections for Yb:YAG.

a) *Calculate the absorption cross section σ_a as a function of the wavelength by dividing the absorption by the reference signal and using Lambert Beer's law and plot your results. The crystal has a 2 at. % ($2{,}74 \cdot 10^{20}\,\frac{1}{cm^3}$) doping concentration and 5 mm thickness.*



b) *In the sketch on the right side you can see the energy levels of Yb:YAG. The energies are given in inverse centimetres ($1/\lambda$) relative to the ground state. Calculate the energy of all possible electronic transitions in eV and compare your results with the absorption peaks in a). Mark the position of the zero phonon line.*

c) *Use the energy levels from b) to calculate the partion functions $Z_{u/l}$ as given in the lecture. The temperature is 20 °C and all degneracies d_i are equal to 2.*

d) *Calculate the emission cross section σ_e with the help of the McCumber relation using the absorption cross section from a) and the partion functions as well as the zero phonon energy. Plot it in the same diagram as σ_a.*

e) *Given the emission spectrum from "fluorescence.csv", the refractive index $n = 1.8$ and the radiative life time $\tau_{rad} = 0{,}95\,\mathrm{ms}$ we can also calculate the emission cross section from the Füchtbauer-Ladenburg equation, without the help of McCumber. For this we need to calculate the real line shape function $g(\nu)$ out of the fluorescence spectrum using the formula*

$$g_\lambda(\lambda) = \frac{\frac{\lambda^3}{c} I_f(\lambda)}{\int_{\lambda_{min}}^{\lambda_{max}} \lambda' I_f(\lambda')\,\mathrm{d}\lambda'} \tag{2.1}$$

where I_f can be directly taken as the fluorescence data and λ_{min} and λ_{max} are minimal and maximal measured wavelengths. Calculate the resulting cross section again and compare it with the one of e).

a.) Solution:

Lambert Beer's law states that the intensity in a medium with absorption coefficient α is reduced exponentially as follows

$$I(z) = I_0 \exp(-\alpha z), \tag{2.2}$$

where $\alpha = N \cdot \sigma_a$. We can now obtain the absorption coefficient by solving this formula for σ_a

$$\sigma_a = \frac{1}{N \cdot z} \ln\left(\frac{I_0(\lambda)}{I(z,\lambda)}\right), \tag{2.3}$$

where $I_0(\lambda)$ describes the reference spectrum and $I(z,\lambda)$ the absorption spectrum. Using this we obtain a graph displayed in figure 1.

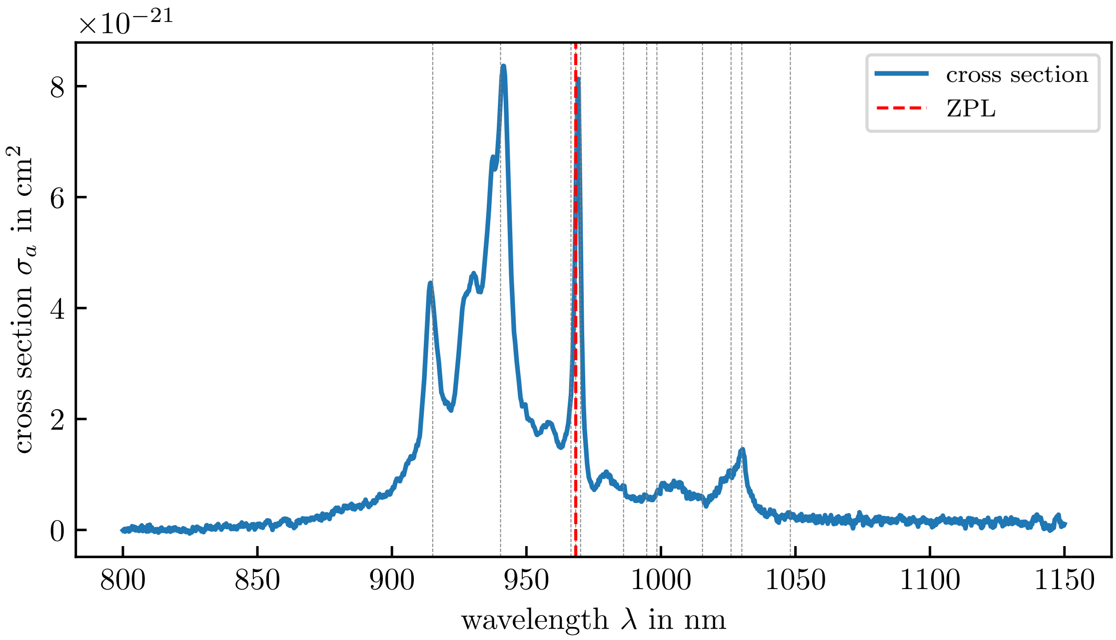


Fig. 1: Absorption cross section σ_a of Yb:YAG as a function of wavelength. The gray dashed lines represent the different electronic transitions. The zero phonon line is indicated in red.

b.) Solution:

We can translate the energies given as inverse wavelengths as follows:

$$E = \frac{h \cdot c}{\lambda} = 1{,}24 \cdot 10^{-4}\,\text{eV} \cdot \frac{1}{\lambda}[\frac{1}{\text{cm}}]. \tag{2.4}$$

Then we can calculate the energy of all twelve possible electronic transitions:

Table 1: All possible electronic transitions in Yb:YAG. The transition occurs from the upper level (u) to the lower level (l).

Transition	Energy [ev]	wavelength [nm]
u0 → l0	1.281	968.3
u1 → l0	1.319	940.4
u2 → l0	1.355	915.2
u0 → l1	1.209	1026.1
u0 → l2	1.204	1030.1
u0 → l3	1.183	1048.1
u1 → l1	1.247	994.7
u1 → l2	1.242	998.5
u1 → l3	1.221	1015.4
u2 → l1	1.283	966.6
u2 → l2	1.278	970.2
u2 → l3	1.257	986.1

The transitions are marked in figure 1 as gray lines.

c.) Solution:

Now we calculate the partition functions Z_u and Z_l of the upper and lower laser levels. We assume a thermal equilibrium at 293 K ($k_B T = 26\,\text{meV}$). The partition function is then given as

$$Z = \sum_k d_k \exp\left(-\frac{E_k}{k_B T}\right), \quad \text{where} \quad d_k = 2. \tag{2.5}$$

We need to keep in mind, that for the upper laser level, the energy difference to the lowest level at $10\,327\,\frac{1}{\text{cm}}$ matters and not the energy with respect to the ground state. The calculation gives us a value of

$$\frac{Z_l}{Z_u} = 0.8843. \tag{2.6}$$

d.) Solution:

Now we can use the McCumber relation to obtain the emission cross section. Using that $E_{\mathrm{ZPL}} = 1{,}28\,\mathrm{eV}$ we can calculate σ_e as

$$\sigma_e = \frac{Z_l}{Z_u} \exp\left(\frac{E_{\mathrm{ZPL}} - h\nu}{k_B T}\right) \sigma_a. \tag{2.7}$$

The calculated spectrum is shown in figure 2.

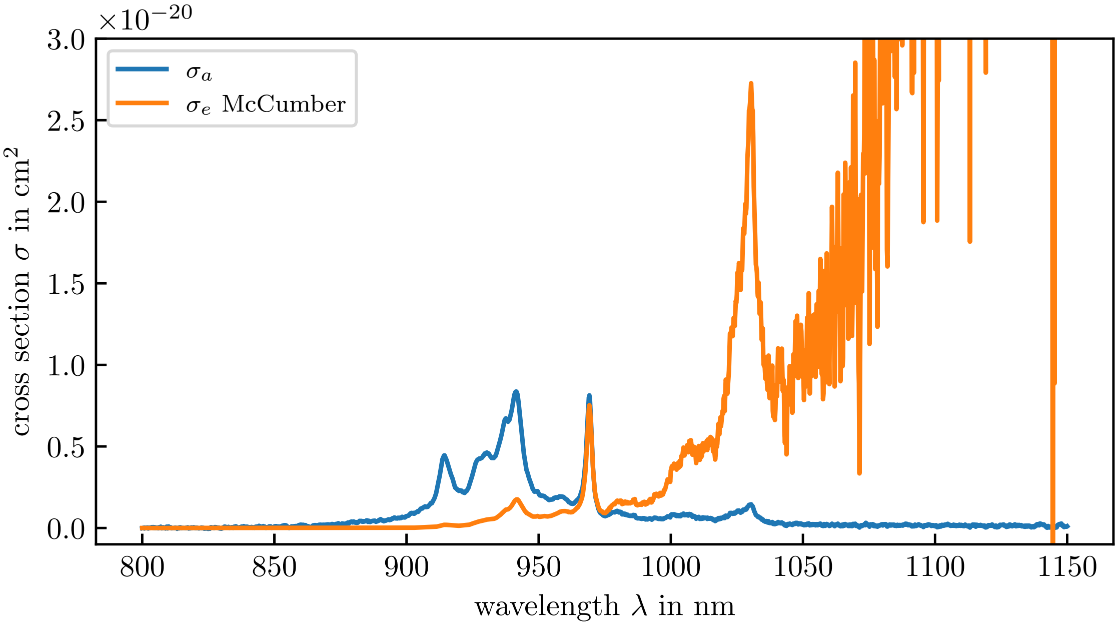


Fig. 2: Emission cross section σ_a (orange) of Yb:YAG as a function of wavelength calculated by the McCumber relation.

We observe that the calculation gets problematic for high wavelength with low absorption cross sections. We conclude that all values above $\lambda = 1050\,\mathrm{nm}$ suffer from this problem. However, we can see that the emission is very high at the characteristic line of 1030 nm which is mostly used for Yb:YAG lasers (like POLARIS).

e.) Solution:

Now we proceed to find another way to calculate the emission spectrum, especially for longer wavelengths. For that we use the Füchtbauer-Ladenburg equation

$$\sigma_e = \frac{\lambda^2}{8\pi n^2 \tau_{\mathrm{rad}}} g(\lambda, \lambda_0). \tag{2.8}$$

The line function $g(\lambda, \lambda_0)$ can be obtained via the fluorescence measurement provided in the task. Here we simply implement equation (2.1) to our program. The given spectrum was integrated numerically using the simpson rule. The resulting spectrum is displayed in figure 3.

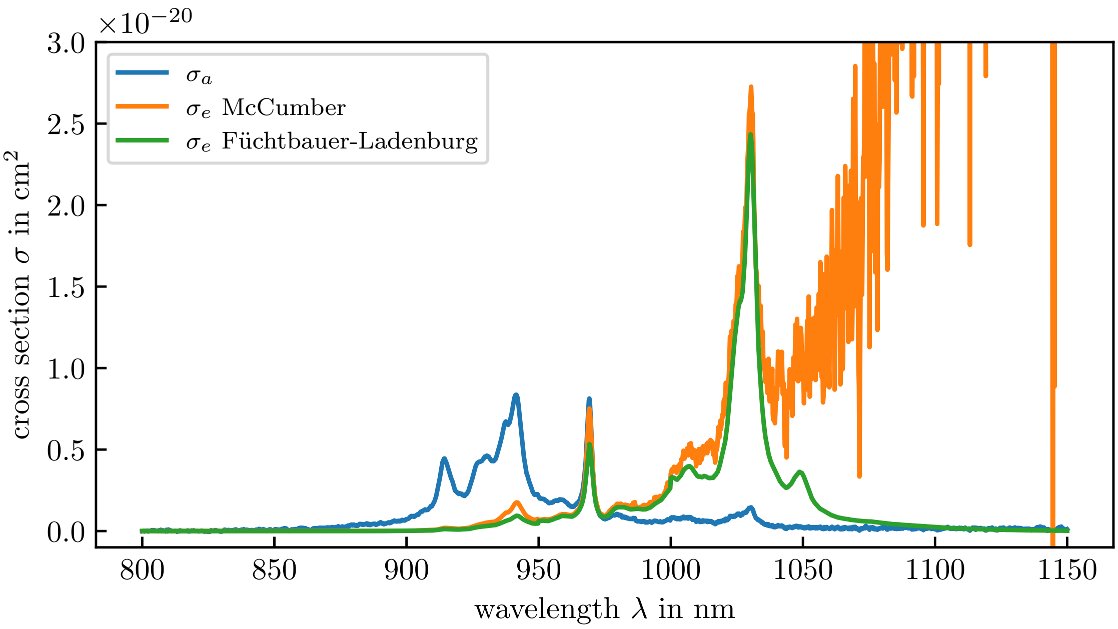


Fig. 3: Emission cross section σ_a (orange) of Yb:YAG as a function of wavelength calculated by the Füchtbauer-Ladenburg equation.

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy.integrate as integrate

absorption = np.loadtxt("absorption.csv", delimiter=",")
reference = np.loadtxt("reference.csv", delimiter=",")
flourescence = np.loadtxt("flourescence.csv", delimiter=",")

#%%
N = 2.74e20    # Doping concentration in cm^-3
d = 0.5        # Thickness in cm
hc = 1.24e-4   # planck constant * speed of light per 1cm in [eV]
kbT = 0.025266 # Energy of room temperature

sigma_a = np.log(reference[:,1]/absorption[:,1])/(N*d)

#%% calculate energies
inverse_lambdas = np.asarray([581, 619, 786, 10327, 10634, 10927])
energies = hc*inverse_lambdas

transitions = energies[3:]

# loop for every transition
for i in range(3):
    for j in range(3):
        transitions = np.append(transitions,
        transitions[i]-energies[j])

transition_lambda = hc/transitions*1e7

#%% Mc-Cumber Relation for emission cross section
Z_l = 0
Z_u = 0
ZPL = energies[3]

F72 = np.append([0],energies[:3])  # Energies in the F 7/2 level
F52 = energies[3:]-energies[3]     # Energies in the F 5/2 level

for i in range(4):
    Z_l += 2*np.exp(-F72[i]/kbT)           # lower laser level
    if i<3: Z_u += 2*np.exp(-F52[i]/kbT)

# emission cross section
sigma_e = Z_l/Z_u*np.exp((ZPL-hc/(1e-7*reference[:,0]))/kbT)*sigma_a

#%% Fuechtbauer-Ladenburg equation
n   = 1.8      # refractive index
tau = 9.5e-4   # life time in s
c   = 3e17     # speed of light in nm/s
Integral = integrate.simps(flourescence[:,1]*flourescence[:,0],
                                    flourescence[:,0])

# cross section: calculated in 1/nm^2 --> multiply by 1e-14 for 1/cm^2
sigma_e2 = flourescence[:,0]**5/(8*np.pi*n*tau*c)*flourescence[:,1]/
                                    Integral*1e-14
```

3 Problem sheet 3

3.1 Exercise 3 - Two level system

Assume a two level system where the degeneracies of the upper and lower level are identical. Set up a rate equation containing absorption, spontaneous emission and stimulated emission. How big is the achieved density of the excited ions at a given radiation energy density for times much longer than the radiative lifetime? Discuss why the lifetime is an important parameter for diode pumping.

Solution:

The rate of change of the density of ions in the excited state N_2 is determined by absorption (B_{12}), stimulated emission (B_{21}) and spontaneous emission (A_{21}). Since the degeneracies are identical we find

$$B_{21} = \frac{g_1}{g_2} B_{12} = B_{12}. \tag{3.1}$$

In the rate equation, emission is proportional to the density of excited ions N_2, while absorption is proportional to the density of ions in the ground state N_1. Additionally we use that the total density of doped ions $N_0 = N_1 + N_2$ is constant. Then the rate equations can be written as

$$\begin{aligned} -\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{\mathrm{d}N_2}{\mathrm{d}t} &= -\rho B_{21} N_2 + \rho B_{12} N_1 - A_{21} N_2 \\ &= B_{21}\rho(N_1 - N_2) - A_{21} N_2 \\ &= -(A_{21} + 2B_{21}\rho) N_2 + N_0 \rho B_{21}. \end{aligned} \tag{3.2}$$

For times much longer than the radiative lifetime, spontaneous emission cannot be neglected. We solve this inhomogeneous differential equation in the following. Starting with the homogeneous solution we find

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -(A_{21} + 2B_{21}\rho) N_2 \quad \Rightarrow N_2(t) = U(t) \exp\bigl(-(A_{21} + 2B_{21}\rho)t\bigr). \tag{3.3}$$

Now using the variation of constants we obtain

$$\begin{aligned} \frac{\mathrm{d}N_2}{\mathrm{d}t} = \frac{\mathrm{d}U}{\mathrm{d}t} \mathrm{e}^{-(A_{21}+2B_{21}\rho)t} - \cancel{(A_{21} + 2B_{21}\rho) N_2(t)} &\overset{!}{=} -\cancel{(A_{21} + 2B_{21}\rho) N_2(t)} + N_0 \rho B_{21} \\ \Rightarrow \frac{\mathrm{d}U}{\mathrm{d}t} &= N_0 \rho B_{21} \mathrm{e}^{(A_{21}+2B_{21}\rho)t} \\ \Rightarrow U(t) &= \frac{N_0 \rho B_{21}}{A_{21} + 2B_{21}\rho} \mathrm{e}^{(A_{21}+2B_{21}\rho)t} + C. \end{aligned} \tag{3.4}$$

Then we find the general solution as

$$N_2(t) = \left(\frac{N_0 \rho B_{21}}{A_{21} + 2B_{21}\rho} \mathrm{e}^{(A_{21}+2B_{21}\rho)t} + C \right) \exp\bigl(-(A_{21} + 2B_{21}\rho)t\bigr). \tag{3.5}$$

The lifetime is an important parameter since it determines the maximum population in the excited state. Using $N_2(t=0)=0$ we find a special solution

$$N_2(t) = \frac{N_0 \rho B_{21}}{A_{21} + 2B_{21}\rho}\left(1 - \exp\big(-(A_{21} + 2B_{21}\rho)t\big)\right). \tag{3.6}$$

The maximum density of the excited state is then given as

$$N_{2,\max} = \frac{N_0 \rho B_{21}}{A_{21} + 2B_{21}\rho}. \tag{3.7}$$

We observe that the higher the spontaneous emission, the lower the maximum population in the excited state.

3.2 Exercise 4 - Emission cross section

How much is the peak emission crosss section of a laser material exhibiting a single Lorentzian line with a bandwidth of 5 nm FWHM. The refractive index of the material is 1.85, the central wavelength 1,03 µm and the radiative lifetime 950 µs. Compare the result with Yb:YAG.

Solution:

We can write down the normalized line function of the Lorentzian as

$$g(\nu, \nu_0) = \frac{2}{\pi \Delta\nu} \frac{1}{1 + \dfrac{(\nu - \nu_0)^2}{4\Delta\nu^2}}, \tag{3.8}$$

where $\Delta\nu$ is the freqeuncy bandwidth at 1030 nm calculated as

$$\Delta\nu = c\left(\frac{1}{\lambda - \Delta\lambda/2} - \frac{1}{\lambda + \Delta\lambda/2}\right) = c\left(\frac{1}{1\,1027{,}5\,\text{nm}} - \frac{1}{1\,1032{,}5\,\text{nm}}\right) = 1{,}41 \cdot 10^{12}\,\text{Hz}. \tag{3.9}$$

d We can now use the Füchtbauer-Ladenburg equation to calculate the cross section at ν_0

$$\sigma_e = \frac{\lambda^2}{8\pi n^2 \tau_{\text{rad}}} g(\nu_0) = \frac{1030\,\text{nm}}{8\pi(1.85)^2 \cdot 950\,\mu\text{s}} \frac{2}{\pi \cdot 1{,}41 \cdot 10^{12}\,\text{Hz}} = 5{,}862 \cdot 10^{-20}\,\text{cm}^2. \tag{3.10}$$

If we compare this with the emission cross section of Yb:YAG we find, that both values are in the same order of magnitude (c.f. figure 4)

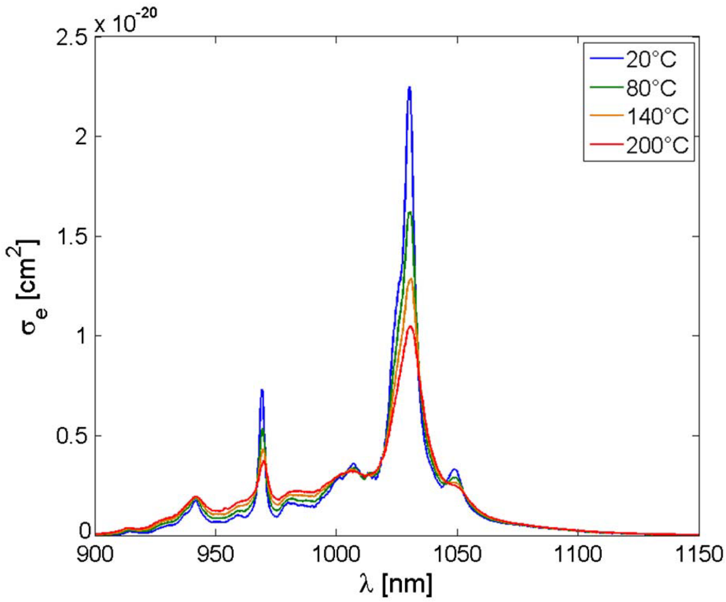


Fig. 4: Emission cross sections of YB:YAG for different temperatures.
Koerner et. al, Measurement of temperature-dependent absorption and emission spectra of Yb:YAG, J.Opt.Soc.Am.B, 2012

3.3 Exercise 5 - Heat fraction

YB:CaF$_2$ has two interesting spectral lines at 991 nm and 995 nm in its emission spectrum. Using the zero phonon line (980,5 nm) for pumping the achievable quantum defect can be extremely low.

- **a)** *What would be the heat fraction of such a laser?*
- **b)** *What is the minimum fraction of excited ions to operate such a laser? What is the maximum fraction you could achieve? Determine these values for 300 K and 80 K. Discuss the results.*
- **c)** *In an actual laser of this kind one measures a considerable higher heat fraction. Assuming unity quantum efficiency, what could be the origin?*

a.) Solution

The heat fraction is the amount of energy lost between the pump energy and laser energy. It is given by

$$\eta = 1 - \frac{E_{\text{laser}}}{E_{\text{pump}}} = 1 - \frac{\nu_l}{\nu_p} = 1 - \frac{\lambda_p}{\lambda_l}. \tag{3.11}$$

For $\lambda_l = 991\,\text{nm}$ we find

$$\eta = 1 - \frac{980.5}{991} = 0.0106 = 1{,}06\,\%. \tag{3.12}$$

For $\lambda_l = 995\,\text{nm}$ we have $\eta = 1{,}45\,\%$. Realistic heat fractions, however, are in the order of 20 to 40 %.

b.) Solution

We start with the McCumber relation for emission and absorption cross section

$$\frac{\sigma_e}{\sigma_a} = \frac{Z_L}{Z_U} \exp\left(\frac{E_{\mathrm{ZP}} - h\nu}{k_B T}\right). \tag{3.13}$$

Now we can calculate the inversion β_{eq} for this using the relation between σ_e and σ_a for an operation at 991 nm

$$\beta_{\mathrm{eq}} = \frac{f_1}{f_1 + f_2} = \frac{1}{1 + \frac{Z_L}{Z_U} \exp\left(\frac{E_{\mathrm{ZP}} - h\nu}{k_B T}\right)} = \frac{1}{1 + 1.678} = 37\,\%. \tag{3.14}$$

β_{eq} is the minimal inversion to operate a laser. Analogously, for 995 nm we find $\beta_{\mathrm{eq}} = 33\,\%$. Thus, a higher λ is better for laser operation but it has a higher heat fraction η. The minimal inversion is lower for higher wavelengths, since the lower laser level is on a higher energy and thus thermally less populated. For smaller temperatures the non radiative decay happens quicker and spontaneous emission plays a smaller role. Thus we find the following values:

$$\beta_{\mathrm{eq}} = \frac{1}{1 + \frac{Z_L}{Z_U} \exp\left(\frac{E_{\mathrm{ZP}} - h\nu}{k_B T}\right)} = \begin{cases} 12{,}5\,\% & \lambda_l = 991\,\mathrm{nm} \\ 6{,}5\,\% & \lambda_l = 995\,\mathrm{nm} \end{cases}. \tag{3.15}$$

The maximum fraction of ions in the excited state is given by the equilibrium inversion at the pump wavelength. Since we are pumping at the zero phonon line we find

$$\beta_{\mathrm{eq,pump}} = \frac{1}{1 + \frac{Z_L}{Z_U} \exp\left(\frac{E_{\mathrm{ZP}} - E_{\mathrm{ZP}}}{k_B T}\right)} = \frac{1}{2}. \tag{3.16}$$

The maximum inversion we can achieve (for infinity pump intensity) is 50 %, which makes sense because at the ZPL absorption and emission cross sections are equal.

c.) Solution

Suppose we achieve an inversion of 40 % in our laser medium and we want to operate the laser at 991 nm. Then the effective share of excited ions used for the lasing process is 3 %. The rest of the excited ions will not be used and can potentially decay via non-radiative processes or via spontaneous emission.

3.4 Exercise 6 - Lifetimes

You analyzed a novel laser material. From the spectral measurements you estimated the radiative lifetime to be 15 ms. A direct measurement of the fluorescence lifetime results in 10 ms.

a) *Explain how you could have determined the radiative lifetime from absorption and emission measurements.*

b) *Assuming, that the fluorescence lifetime is only influenced by additional non-radiative decay, calculate the quantum efficiency and the time constant of the non-radiative decay.*

c) *Considering only spontaneous emission, how much heat is generated in this material when it is pumped with 10 W (the heat fraction due to the quantum defect is 15 %).*

a.) Solution

One way to determine the radiative lifetime is to first determine the line shape function $g(\nu,\nu_0)$ via fluorescence measurements. Then we can determine the emission cross section using the absorption cross sections and the McCumber relation. Finally we find that the lifetime is given by the Füchtbauer-Ladenburg equation as

$$\tau_{\text{rad}} = \frac{c_0^2}{8\pi n^2 \sigma_e \nu^2} g(\nu,\nu_0). \tag{3.17}$$

Alternatively if the emission cross section is already known we can use the following formula:

$$\frac{1}{\tau_{\text{rad}}} = \frac{8\pi n^2}{c^2} \int \nu^2 \sigma_e(\nu)\,\mathrm{d}\nu. \tag{3.18}$$

b.) Solution

We obtain the fluorescence life time by adding the probabilities of the radiative decay and non-radiative decay

$$\frac{1}{\tau_f} = \frac{1}{\tau_{\text{rad}}} + \frac{1}{\tau_{\text{nr}}} \quad \Rightarrow \quad \tau_{\text{nr}} = \frac{\tau_{\text{rad}} \cdot \tau_f}{\tau_{\text{rad}} - \tau_f} = 30\,\text{ms}. \tag{3.19}$$

The quantum efficiency is then given as

$$\eta_{QE} = \frac{\int_0^\infty \mathrm{e}^{-t/\tau_{\text{rad}}}}{\int_0^\infty \mathrm{e}^{-t/\tau_{\text{rad}}} + \int_0^\infty \mathrm{e}^{-t/\tau_{\text{nr}}}} = \frac{\tau_{\text{rad}}}{\tau_{\text{rad}} + \tau_{\text{nr}}} = 66\,\%. \tag{3.20}$$

c.) Solution

The generated heat in this material is

$$P_{\text{heat}} = P_0(1-\eta_{QE}) + P_0\eta_{QE}\eta_{QD} = P_0(1-\eta_{QE}+\eta_{QE}\eta_{QD}) = 4{,}4\,\text{W}. \tag{3.21}$$

4 Problem sheet 4

4.1 Exercise 7 - Amplification

Assume a laser material with a constant cross section from 1000 nm…1080 nm. Otherwise the emission cross section is zero in the emission band. The zero phonon line is at 990 nm. Determine in which wavelength range amplification could be achieved if 30 % of the ions are excited. Compare the results at 300 K and 80 K.

Solution:

Assuming that 30 % of the ions are excited we have for $N_2 = 0.3N_{\text{dop}}$ and thus $\beta = 0.3$. According to the solution for the photon density given in the lecture

$$\Phi = \Phi_0 \exp\left(\sigma_e N_{\text{dop}} f_2 \frac{f_2 + f_1}{f_2} (\beta - \beta_{\text{eq}}) z\right), \tag{4.1}$$

we only have gain if $\beta > \beta_{\text{eq}}$. Now in order to determine the wavelength range in which amplification is possible we calculate the equilibrium inversion

$$\beta_{\text{eq}} = \frac{f_2}{f_1 + f_2} = \frac{\sigma_a}{\sigma_a + \sigma_e} \tag{4.2}$$

using the McCumber relation where we can write the absorption cross section σ_a as

$$\frac{\sigma_e}{\sigma_a} = \exp\left(\frac{E_{\text{ZL}} - h\nu}{k_B T}\right) \quad \Rightarrow \quad \sigma_a = \sigma_e \exp\left(\frac{h\nu - E_{\text{ZL}}}{k_B T}\right). \tag{4.3}$$

Then the equilibrium inversion becomes

$$\beta_{\text{eq}} = \frac{\exp\left(\frac{h\nu - E_{\text{ZL}}}{k_B T}\right)}{\exp\left(\frac{h\nu - E_{\text{ZL}}}{k_B T}\right) + 1}. \tag{4.4}$$

We can plot this function for the two different temperatures as shown in figure 5.

As we can see the equilibrium inversion is smaller than the inversion in the wavelength range of 1008 nm…1080 nm. For lower the lower temperature the equilibrium inversion is always smaller than 0.3.

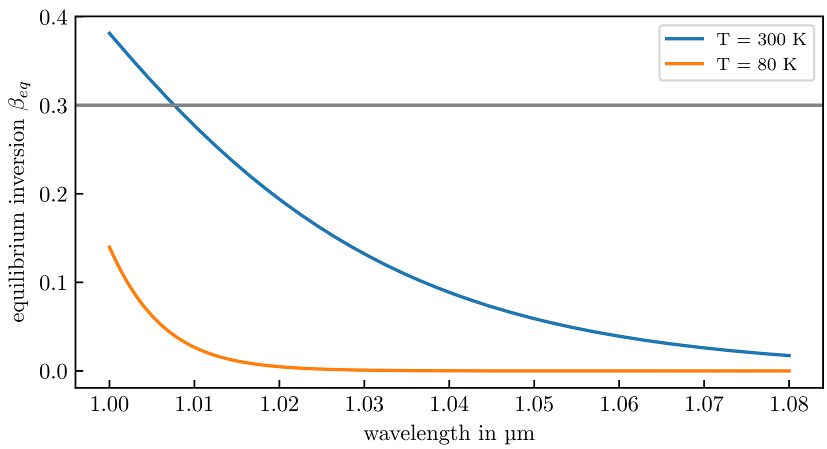


Fig. 5: Equilibrium inversion β_{eq} for the wavelengths of nonzero emission cross section. The gray horizontal line indicates the inversion of the laser material.

4.2 Exercise 8 - Amplifier design

You are to design an amplifier based on Yb:YAG with 3 atomic percent doping with an end pump layout. The pump duration corresponds to τ_f.

a) *In order to utilize as much of your pump light as possible, you need to achieve an inversion of at least $\beta_{eq}(\lambda_l)$, where $\lambda_l = 1030\,\mathrm{nm}$ corresponds to your laser wavelength. At which point of the crystal will this inversion be reached? How big is it, if the laser material is operated at 300 K?*

b) *You plan to use a pump fluence of 30 $\frac{kW}{cm^2}$ with a pump source matching the absorption peak close to 940 nm. How thick got to be the laser crystal under the constrictions given in part a) if you assume Lambert Beer's law? Compare this value with the result obtained by the simulation code.*

c) *What is the inversion at the input surface of the crystal in dependence of time?*

d) *Use the simulation to calculate gain and efficiency of this configuration? How do parameters change if you increase/decrease the pump intensity? What limitations do you see?*

Note: *Please use the supplied simulation to solve parts of this exercise. If you increase the pump duration, make sure to adjust the numerical resolution accordingly.*

a.) Solution:

We start by calculating $\beta_{\mathrm{eq}}(\lambda_l)$ using (4.2)

$$\beta_{\mathrm{eq}}(1030\,\mathrm{nm}) = \frac{\sigma_a}{\sigma_a + \sigma_e} = \frac{0{,}12 \cdot 10^{-20}\,\mathrm{cm}^2}{0{,}12 \cdot 10^{-20}\,\mathrm{cm}^2 + 2{,}2 \cdot 10^{-20}\,\mathrm{cm}^2} = 0.0517. \tag{4.5}$$

We use the provided program to calculate the inversion. For the density of dopant ions we use the fact that the density of Yb at 1 atomic % is given as $N_{\mathrm{dop}} = 1{,}38 \cdot 10^{20}\,\frac{1}{\mathrm{cm}^3}$[1].

We use a pump duration of $t_{\mathrm{pump}} = \tau_f = 950\,\mu\mathrm{s}$ and a pump intensity of $30\,\frac{\mathrm{kW}}{\mathrm{cm}^2}$ at 940 nm. The simulation provides us with a graph displayed in figure 6. We observe that $\beta_{\mathrm{eq}}(\lambda_l)$ is reached at about 9 mm into the crystal.

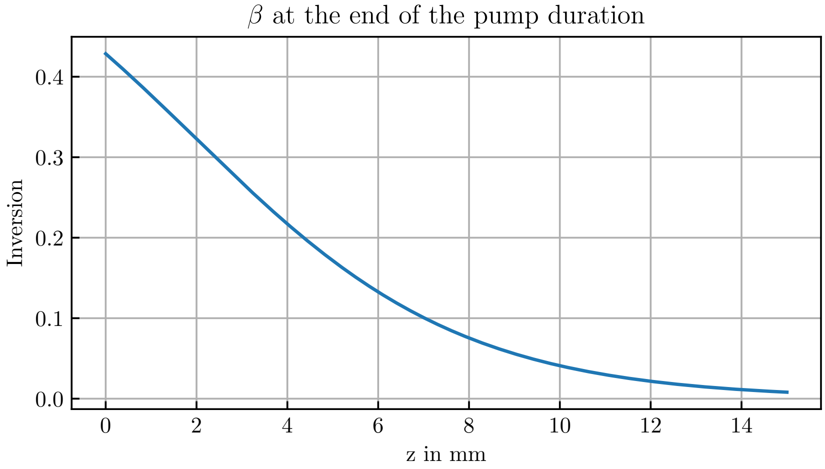


Fig. 6: inversion $\beta(z)$ at the end of pumping in the crystal. The gray horizontal line indicates the inversion of the laser material.

b.) Solution:

If we now assume Lambert-Beer's law we can simply find the inversion in the following way:

$$\beta(z) = \beta_0 \exp\left(-N_{\mathrm{dop}}\sigma_a z\right), \tag{4.6}$$

where $\beta_0 = 0.4286$ was found using the first value of the plot in figure 6. Using $N_{\mathrm{dop}} = 4{,}12 \cdot 10^{20}\,\frac{1}{\mathrm{cm}^3}$ and $\sigma_a = 0{,}82 \cdot 10^{-20}\,\mathrm{cm}^2$ we can plot this as shown in figure 7.

We also try to find the value of β_0 without using the simulation. For that we assume a steady-state condition as in chapter III.2.1. Here we found for the pump rate

$$R = \frac{\beta}{f_p \tau_f \left(1 - \frac{\beta}{\beta_{\mathrm{eq}}}\right)}. \tag{4.7}$$

We also note that the pump rate is connected to the pump intensity as

$$R = \sigma_a \Phi c = \sigma_a \frac{I}{h\nu}. \tag{4.8}$$

[1] https://www.rp-photonics.com/yag_lasers.html

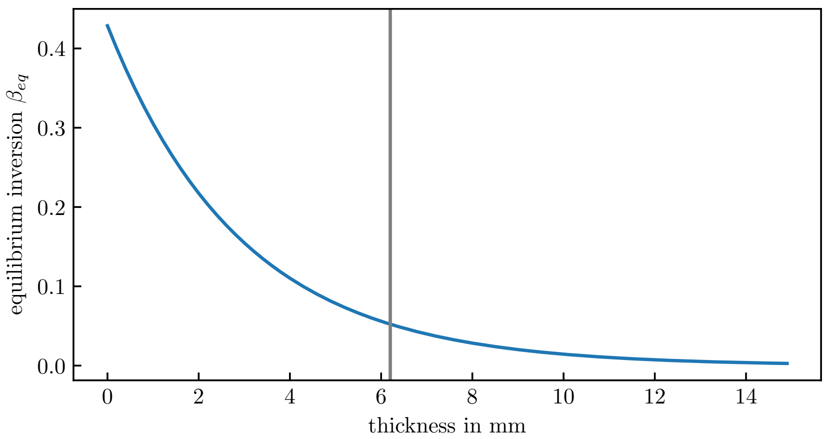


Fig. 7: Inversion $\beta(z)$ at the end of pumping in the crystal if we assume Lambert-Beer's law.

We can rearange (4.7) to find

$$\xi := \frac{I\sigma_a\tau_f f_p}{h\nu} = \frac{\beta}{1-\frac{\beta}{\beta_{\mathrm{eq}}}} \quad \Rightarrow \beta = \frac{\xi}{1+\frac{\xi}{\beta_{\mathrm{eq}}}}. \tag{4.9}$$

Using $f_p = 1$ (which is reasonable since the pump process typically starts at the lowest level of the lowest manifold), $\tau_f = 950\,\mu\mathrm{s}$, $I = 30\,\frac{\mathrm{kW}}{\mathrm{cm}^2}$ we find with

$$\beta_{\mathrm{eq}}(940\,\mathrm{nm}) = \frac{\sigma_a}{\sigma_a+\sigma_e} = \frac{0{,}82\cdot 10^{-20}\,\mathrm{cm}^2}{0{,}82\cdot 10^{-20}\,\mathrm{cm}^2 + 0{,}17\cdot 10^{-20}\,\mathrm{cm}^2} = 0.8283 \tag{4.10}$$

a value for β_0

$$\xi = 1.11 \quad \Rightarrow \quad \beta_0 = \frac{1.11}{1+\frac{1.11}{0.8283}} = 0.475, \tag{4.11}$$

which corresponds to the inversion reached in the limit of the pump time going to infinity. Since we only pump in the order of the fluorescence life time, the inversion at face of the crystal is lower than expected.

c.) Solution:

In figure 8 we can see the inversion at the input surface as a function of time. This result was also obtained by modifying the simulation code.

d.) Solution

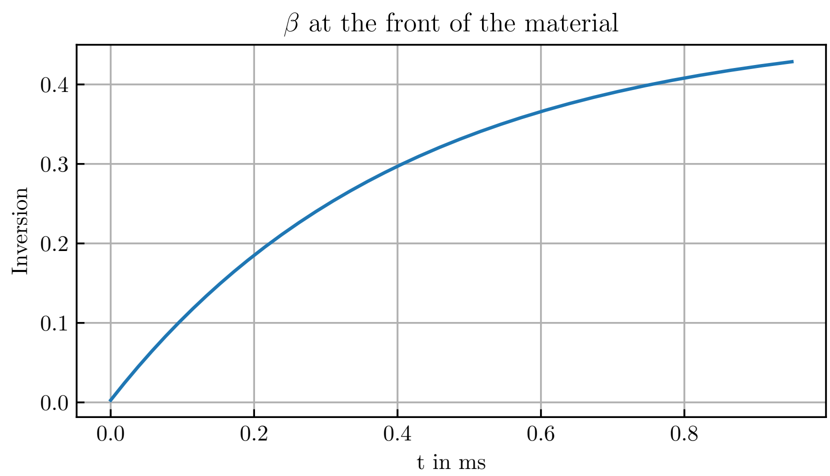


Fig. 8: Inversion $\beta(t)$ at the front face of the crystal as a function of time.

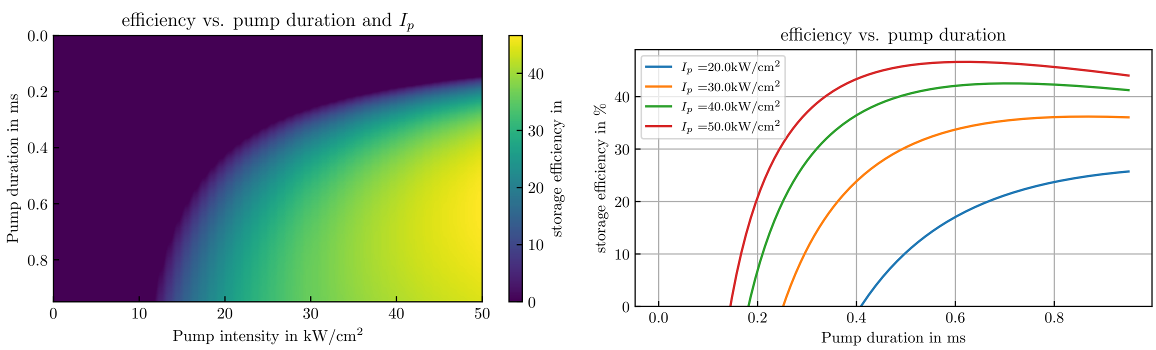


Fig. 9: Left: Plot of the storage efficiency as a function of the pump duration for various pump intensities. Right: Vertical slices of the left image for different pump intensities.

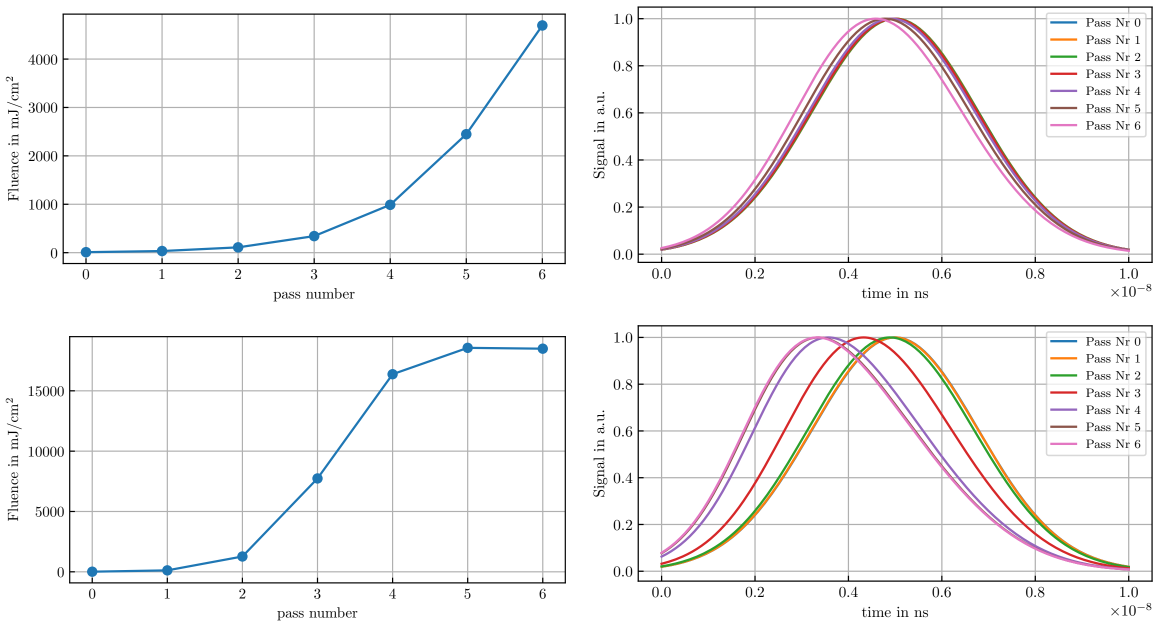


Fig. 10: Left: Laser fluence as a function of the number of passes through the amplifying medium. Right: Time signal of the amplified pulse for the various pass numbers.
The top row shows calculations for $I_P = 30\,\frac{\text{kW}}{\text{cm}^2}$ while the bottom row shows calculations for $I_P = 50\,\frac{\text{kW}}{\text{cm}^2}$.

5 Problem sheet 5

5.1 Exercise 8 - Opposite matrix

The matrix of an optical system is given as $M_{\rightarrow} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. *Prove that the matrix for passing the system in opposite is* $M_{\leftarrow} = \begin{pmatrix} D & B \\ C & A \end{pmatrix}$.

Solution:

Proof. We start by noting, that the transfer matrix for passing the system in opposite direction is not the inverse of the original matrix. If we pass a single element, like a lens, it does not matter, whether of not we pass it from left or right (This is true as far as ray optics is concerned, however, abberations can be different, e.g. in a plane-convex lens). Therefore, for single elements we must observe $M_{\rightarrow} = M_{\leftarrow}$ which is indeed the case:

$$M_{\text{lens}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \quad M_{\text{free space}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \quad M_{\text{mirror}} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix}. \tag{5.1}$$

We can see, that all transfer matrices have the following structure:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}. \tag{5.2}$$

For a system of two elements we can easily calculate $M_{\rightarrow}$ and $M_{\leftarrow}$. Remember, that for N-elements the transfer matrix is given by

$$M_{\rightarrow} = M_N \cdot M_{N-1} \ldots M_2 \cdot M_1, \qquad M_{\leftarrow} = M_1 \cdot M_2 \cdot M_{N-1} \ldots M_N. \tag{5.3}$$

For a system of two transfer matrices we find

$$\begin{aligned} M_{\rightarrow} = M_2 \cdot M_1 &= \begin{pmatrix} 1 & C \\ D & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ B & 1 \end{pmatrix} = \begin{pmatrix} 1+AD & A+C \\ B+D & 1+BC \end{pmatrix}, \\ M_{\leftarrow} = M_1 \cdot M_2 &= \begin{pmatrix} 1 & A \\ B & 1 \end{pmatrix} \begin{pmatrix} 1 & C \\ D & 1 \end{pmatrix} = \begin{pmatrix} 1+BC & A+C \\ B+D & 1+AD \end{pmatrix}. \end{aligned} \tag{5.4}$$

We can see that these matrices fulfill the required condition. We can now try to generalize this for N-matrices. We start with three matrices first:

$$\begin{aligned} M_{\rightarrow} = M_3 \cdot M_2 \cdot M_1 &= \left[\mathbb{1} + \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix} \right] \cdot \left[\mathbb{1} + \begin{pmatrix} 0 & A+C \\ B+D & 0 \end{pmatrix} + \begin{pmatrix} AD & 0 \\ 0 & BC \end{pmatrix} \right] \\ &= \mathbb{1} + \begin{pmatrix} 0 & A+C \\ B+D & 0 \end{pmatrix} + \begin{pmatrix} AD & 0 \\ 0 & BC \end{pmatrix} + \begin{pmatrix} (B+D)E & 0 \\ 0 & (A+C)F \end{pmatrix} + \begin{pmatrix} 0 & BCE \\ ADF & 0 \end{pmatrix}. \end{aligned} \tag{5.5}$$

$$\begin{aligned} M_{\leftarrow} = M_1 \cdot M_2 \cdot M_3 &= \cdot \left[\mathbb{1} + \begin{pmatrix} 0 & A+C \\ B+D & 0 \end{pmatrix} + \begin{pmatrix} BC & 0 \\ 0 & AD \end{pmatrix} \right] \left[\mathbb{1} + \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix} \right] \\ &= \mathbb{1} + \begin{pmatrix} 0 & A+C \\ B+D & 0 \end{pmatrix} + \begin{pmatrix} BC & 0 \\ 0 & AD \end{pmatrix} + \begin{pmatrix} (A+C)F & 0 \\ 0 & (B+D)E \end{pmatrix} + \begin{pmatrix} 0 & BCE \\ ADF & 0 \end{pmatrix}. \end{aligned} \tag{5.6}$$

We observe, that for three matrices the formula also works, as the diagonal matrices have their entries switched. We can now write every possible matrix $M_\rightarrow$ as

$$M_\rightarrow = \mathbb{1} + \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}. \tag{5.7}$$

Now lets assume we showed that $M_\rightarrow$ and $M_\leftarrow$ fulfill the condition, we can inductively show that $M_{N+1} \cdot M_\rightarrow$ and $M_\leftarrow \cdot M_{N+1}$ also fulfill the condition because

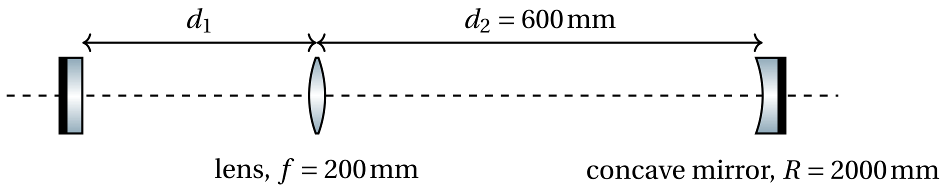
$$\begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} + \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix} \right] = \begin{pmatrix} \beta E & 0 \\ 0 & \alpha F \end{pmatrix} + \begin{pmatrix} 0 & \delta E \\ \gamma F & 0 \end{pmatrix} \tag{5.8}$$

$$\left[\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} + \begin{pmatrix} \delta & 0 \\ 0 & \gamma \end{pmatrix} \right] \cdot \begin{pmatrix} 0 & E \\ F & 0 \end{pmatrix} = \begin{pmatrix} \alpha F & 0 \\ 0 & \beta E \end{pmatrix} + \begin{pmatrix} 0 & \delta E \\ \gamma F & 0 \end{pmatrix}. \tag{5.9}$$

□

5.2 Exercise 9 - Stable resonator with lens

You have a laser resonator as given in the picture below. Calculate for which distances of d_1 this is a stable cavity. What is the maximum mode radius w_0 that can be achieved in this cavity and at which value of d_1 will this be the case? How much pump power do you need if the top-hat shaped pump spot has the same size as the laser modes $1/e^2$ diameter as the mode. The goal pump intensity is 40 $\frac{kW}{cm^2}$.



Solution: In order to find the lengths d_1 for which this cavity is stable we first have to calculate the resonator matrix. The matrix is given as

$$\begin{aligned} M &= \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2d_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-4d_1d_2^2+2Rd_1d_2+Rf^2-4d_1f^2-4d_2f^2-2Rd_1f+4d_2^2f-2Rd_2f+8d_1d_2f}{Rf^2} & \frac{2d_1d_2^2+2d_1f^2+2d_2f^2-2d_2^2f-4d_1d_2f}{f^2} \\ \frac{2Rd_1-4d_1d_2-2f^2-2Rf+4d_1f+4d_2f}{Rf^2} & \frac{2d_1d_2+f^2-2d_1f-2d_2f}{f^2} \end{pmatrix}. \end{aligned} \tag{5.10}$$

In the lecture we derived that the condition for a stable matrix is

$$|A + D| \leq 2, \tag{5.11}$$

which is equivalent to the statement that the q-parameter is complex (which is necessary for a finite beam width).

$$|A+D| = \left| \frac{4d_1d_2}{f^2} - \frac{4d_1}{f} - \frac{4d_2}{f} + 2 - \frac{4d_1d_2^2}{Rf^2} + \frac{8d_1d_2}{Rf} - \frac{4d_1}{R} + \frac{4d_2^2}{Rf} - \frac{4d_2}{R} \right| \leq 2. \tag{5.12}$$

We can solve this equation for d_1 using Mathematica and find

$$175\,\text{mm} \leq d_1 \leq = 300\,\text{mm}. \tag{5.13}$$

Now we also try to find the maximum value of the beam waist and the corresponding distance d_1. For that we start at the beam waist, which lies at the position of the planar mirror (because for a stable resonator the curvature of the beam must match the curvature of the mirror). The q-parameter is then given as $q_{\text{in}} = -\text{i}z_R$ where z_R is the (unknown) Rayleigh length. We can evolve the q-parameter using the transfer matrix to the other mirror

$$\tilde{M} = \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{d_2}{f} & d_2 + d_1\left(1 - \frac{d_2}{f}\right) \\ -\frac{1}{f} & 1 - \frac{d_1}{f} \end{pmatrix}. \tag{5.14}$$

When can now calculate the evolved q-parameter in the following way:

$$q_{\text{out}} = \frac{Aq_{\text{in}} + B}{Cq_{\text{in}} + D} = \frac{d_2 + d_1\left(1 - \frac{d_2}{f}\right) - \text{i}(1 - \frac{d_2}{f})z_R}{1 - \frac{d_1 - \text{i}zR}{f}}. \tag{5.15}$$

We can now obtain an expression for the Rayleigh length z_R by using the fact that the curvature of the phase front is given by $\text{Re}\{1/q\}$, so we can solve the equation

$$\frac{1}{R} = \text{Re}\left\{\frac{1}{q}\right\} = \text{Re}\left(\frac{1 - \frac{d_1 - \text{i}zR}{f}}{d_2 + d_1\left(1 - \frac{d_2}{f}\right) - \text{i}(1 - \frac{d_2}{f})z_R} \right) \tag{5.16}$$

for z_R. Doing that we obtain a rather long expression (assuming z_R is positive)

$$z_R = -\frac{\text{i}\sqrt{d_1d_2 - d_1f - d_2f}\sqrt{d_1d_2 - d_1f - d_2f - d_1R + fR}}{\sqrt{d_2^2 - 2d_2f + f^2 - d_2R + fR}}. \tag{5.17}$$

Now we try to find the maximum value of the Rayleigh length which is achieved for a resonator of length $d_1 = 237{,}5\,\text{mm}$ with $z_R = 62{,}5\,\text{mm}$. We can now use the definition of the Rayleigh length to compute the maximum width of the beam waist

$$z_R = \pi \frac{w_0^2}{\lambda} \quad \Rightarrow \quad w_0 = \sqrt{\frac{\lambda z_R}{\pi}} = 143{,}15\,\mu\text{m} \tag{5.18}$$

assuming a wavelength of 1030 nm. The pump power can now be calculated to

$$P = I_p \pi w_0^2 = 40\,\frac{\text{kW}}{\text{cm}^2} \cdot pi(143{,}15\,\mu\text{m})^2 = 25{,}75\,\text{W}. \tag{5.19}$$

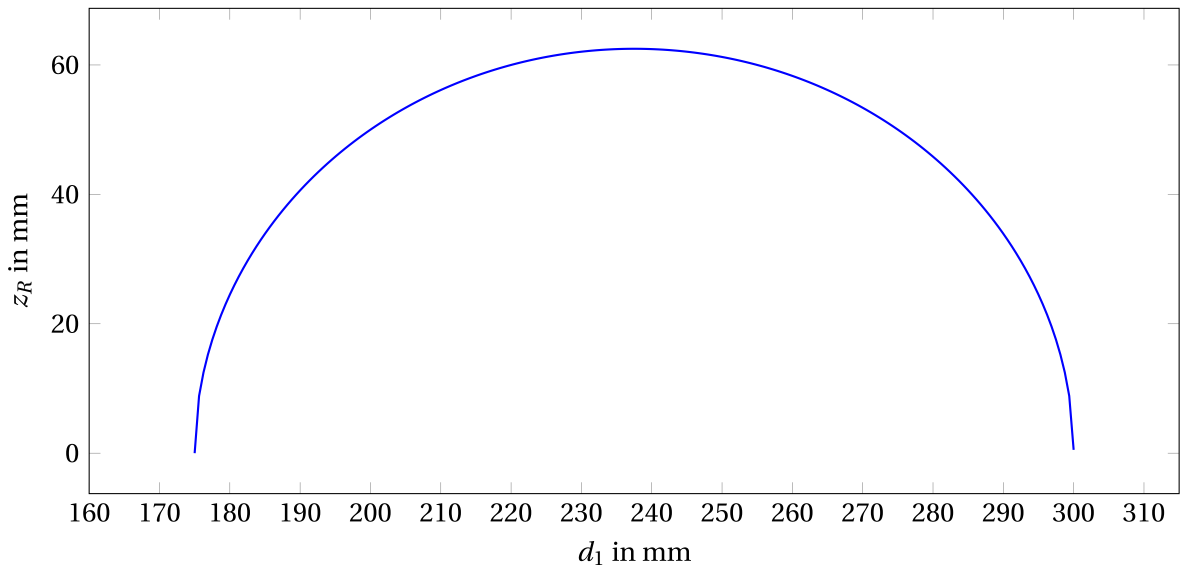


Fig. 11: Plot of the Rayleigh length as a function of d_1. We observe that at the boundaries of stability the Rayleigh length gets zero.

5.3 Exercise 10 - Unstable resonator

In a confocal unstable resonator $R_1 + R_2 = 2L$ is valid, where R_1 and R_2 are the radii of curvature of the end mirrors. Demonstrate that the radius of curvature of the wavefront on the surface of the mirrors is either the corresponding focal length or a plane wave. Furthermore, derive the equation for the magnification in such a cavity.

Solution:

At first we calculate the transfer matrix of the resonator for one round trip starting at mirror R_2 as

$$\begin{aligned}\begin{pmatrix}A & B\\ C & D\end{pmatrix} = M &= \begin{pmatrix}1 & 0\\ -\frac{2}{R_2} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{2}{R_1} & 1\end{pmatrix}\begin{pmatrix}1 & L\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1-\frac{2L}{R_1} & 2L(1-\frac{L}{R_1})\\ -\frac{2}{R_1}\left(1-\frac{2L}{R_2}\right)-\frac{2}{R_2} & 1+L\left(-\frac{2}{R_1}\left(1-\frac{2L}{R_2}\right)-\frac{4}{R_2}\right)\end{pmatrix}\\ &= \begin{pmatrix}-\frac{R_2}{R_1} & \frac{(R_1-R_2)(R_1+R_2)}{2R_1}\\ 0 & -\frac{R_1}{R_2}\end{pmatrix} \quad \text{assuming} \quad 2L = R_1 + R_2.\end{aligned} \tag{5.20}$$

We now follow the course of the lecture where we assume that the radius of curvature should be maintained in an unstable cavity. Within the small angle approximation we have $R \approx \frac{x}{\alpha}$. Using the matrix formalism

$$\begin{pmatrix}x'\\ \alpha'\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ \alpha\end{pmatrix} \tag{5.21}$$

we find

$$\frac{1}{R} \stackrel{!}{=} \frac{1}{R'} = \frac{\alpha'}{x'} = \frac{C + D/R}{A + B/R} \quad \Rightarrow \quad 0 = \frac{B}{R^2} + \frac{A - D}{R} - C. \tag{5.22}$$

Solving this quadratic equation yields

$$\begin{aligned} \frac{1}{R_{1/2}} &= \frac{D-A}{2B} \pm \sqrt{\left(\frac{D-A}{2B}\right)^2 + \frac{C}{B}} \\ &= \frac{1}{2B}(D - A \pm \sqrt{(A+D)^2 - 4}), \end{aligned} \tag{5.23}$$

where in the second line we used $\det M = AD - BC \stackrel{!}{=} 1$ to substitute C. Finally we insert (5.20) into the solution and find by additionally using $2L = R_1 + R_2$

$$\frac{1}{R} = \frac{1}{R_2}\left(-1 \pm \frac{\sigma(R_1)\sigma(R_2)}{\sigma(L)}\right) = \frac{1}{R_2}(-1 \pm 1), \tag{5.24}$$

where σ denotes the signum of the radii and length L. Thus, we find two solutions for the radius of curvature (assuming $|R_1| > |R_2|$)

$$R = \begin{cases} \infty & R_2 > 0 \quad ('+'\text{- solution}) \\ R_2/2 & R_2 < 0 \quad ('+'\text{- solution}) \end{cases}. \tag{5.25}$$

The magnification of the systme is given by

$$M = \frac{x'}{x}, \quad \text{with} \quad x' = Ax + B\alpha. \tag{5.26}$$

Using the definition of R we find

$$x' = \left(A + \frac{B}{R}\right)x \quad \Rightarrow \quad M = A + \frac{B}{R}. \tag{5.27}$$

In the two cases $R_+ = \infty, R_- = -R_2/2$ we find

$$M_+ = A = -\frac{R_2}{R_1}, \quad M_- = A - \frac{2B}{R_2} = -\frac{R_1}{R_2}. \tag{5.28}$$

6 Problem sheet 6

6.1 Exercise 13 - Amplifier design

A collimated laser beam (TEM_{00}) has a diameter of 5 mm. You want to build a Type 1 amplifier using two 2 inch lenses and two plane mirrors. The beam inside the laser crystal shall be one third of the size of your input beam (no external telescope). The total length of the system shall be 2 m at most. For practical reasons the distance between the end mirror and the lens in the input arm has to be 1 m.

- **a)** *Calculate the matrix of the system and make a sketch!*
- **b)** *Where would ou put the laser crystal in the system and how do you want to couple into the system?*
- **c)** *What focal length do the lenses need to have?*
- **d)** *How many material passes can you achieve in the system when the aperture needed on the lens in the input arm is 1 cm per pass? Is the aperture of the second lens sufficient?*
- **e)** *Discuss advantages and disadvantages if you would use spherical mirrors instead of lenses?*
- **f)** *Assume that your laser crystal has a thermal lens with the focal length f_t. How can you compensate for this lens by realigning the system? What values of f_t can you compensate with the system?*
- **g)** *How could an according type 2 system look like?*

a.) Solution:

We can describe the cavity with the matrix formalism. We have to alternatly apply free space propagation and lens matrices to calculate the matrix for a single pass

$$
\begin{aligned}
M_{\rightarrow} &= \begin{pmatrix} 1 & d_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & d_1 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{d_2 d_3}{f_1 f_2} - \frac{d_2+d_3}{f_1} - \frac{d_3}{f_2} + 1 & d_1 + d_2 + d_3 - \frac{d_1(d_2+d_3)}{f_1} - \frac{d_3(d_1+d_2)}{f_2} + \frac{d_1 d_2 d_3}{f_1 f_2} \\ \frac{d_2}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} & 1 - \frac{d_1}{f_1} - \frac{d_1+d_2}{f_2} + \frac{d_1 d_2}{f_1 f_2} \end{pmatrix}.
\end{aligned}
\tag{6.1}
$$

The laser amplifier would be looking like a normal oscillator with two end mirrors with two lenses in between. A sketch is displayed in figure 12.

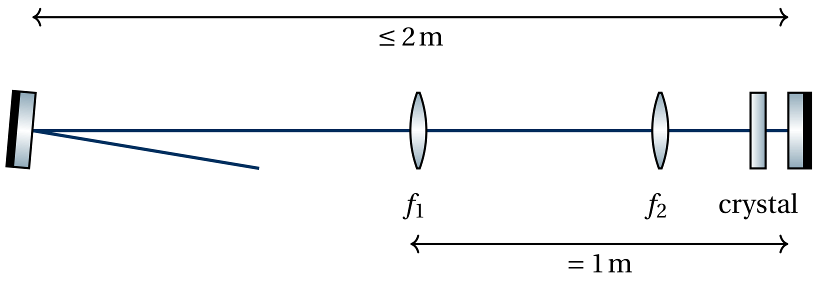


Fig. 12: Type I laser amplifier with lenses.

b.) Solution:

The laser crystal should be placed at one of the end mirrors of our cavity, since there the light is focused. You do not want to put the laser crystal in between the two lenses, since in the focus of the lens we have an image of the input beam from infinity. We would then image potential phase abberations of the input beam into the laser crystal which would not be optimal.

We want to couple into the system under an oblique angle where we slightly tilt the mirror in the input arm. This will lead to several passes through the laser crystal until the beam is coupled out in the same direction as it was coupled in.

c.) Solution:

For a stable solution of the cavity we demand that $M_{\rightarrow}M_{\leftarrow} = \mathbb{1}$. This leads to the condition, that we must place the lenses into a telescope like arrangement meaning that

$$f_1 + f_2 = d_2. \tag{6.2}$$

Since we want to reduce the beam size inside the laser crystal by one third, we choose a magnifying telescope, where the ratio of the focal lengths is $1/3$. For maximum cavity length of $d_1 + d_2 + d_3 = 2\,\mathrm{m}$ we obtain $f_1 = 0{,}525\,\mathrm{m}$ as a stable solution using Mathematica.

d.) Solution:

Every beam propagation through the input lens requires one centimetre. Since we have an aperture of 2 inches (5 cm) we can fit 5 beams through the aperture. In order to decide whether or not the aperture of the second lens is sufficient, we must consider two cases. If the distance between the end mirror and the lens is smaller than the focal length, the beams are diverging behind the lens meaning that the aperture of the second lens is not sufficient. For a larger distance as calculated previously, the beams are converging which means, the aperture of the second lens is big enough.

e.) Solution:

For this task we refer to page eight of the paper of Dr. Körner [2]. The mirror based layout enables smaller lengths scales since the setup can be folded. Additionally, the losses in a mirror based layout are lower than for the corresponding lens design, because we use less optical components. Also the lens based design may generate ”ghost foci“ created by residually reflected light from the lens surfaces which might damage the amplifier. Mirror based designs significantly reduce the collected nonlinear phase (B-integral) and we have less dispersion (no lenses) and thus no chromatic abberations.

However, there is a catch: Mirror based designs suffer more astigmatism than lens based designs, because we have to insert the beam under an additional incident angle in the input because of the large mirrors.

f.) Solution:

The influences of thermal lenses are also described in the paper of Körner. If we have a thermal lens then we have to make sure that the stability condition is preserved

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_{\text{th}}} & \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ -\frac{1}{f_{\text{th}}} A + C & -\frac{1}{f_{\text{th}}} B + D \end{pmatrix} \overset{!}{=} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3}$$

We know that $C = \frac{1}{f_1 f_2}(d_2 - f_1 - f_2)$. Then the new conditions are

$$A \overset{!}{=} -1 \quad \text{and} \quad -\frac{1}{f_{\text{th}}} A + C \overset{!}{=} 0 \quad \text{and} \quad -\frac{1}{f_{\text{th}}} B + D \overset{!}{=} -1. \tag{6.4}$$

Therefore we have

$$\begin{aligned} 0 = \frac{1}{f_{\text{th}}} + C = \frac{1}{f_{\text{th}}} + \frac{1}{f_1 f_2}(d_2 - f_1 - f_2) = \frac{f_1 f_2}{f_{\text{th}}} + d_2 - f_1 - f_2 \\ \Rightarrow d_2 = f_1 + f_2 - \frac{f_1 f_2}{f_{\text{th}}}. \end{aligned} \tag{6.5}$$

g.) Solution:

For a type II system we simply need to set $d_3 = 0$ replace the second lens f_2 with a spherical mirror of focal length $f_2/2$ because it has to act like a double pass of the lens.
In general, one should always choose a type II design because it has less components, is smaller and can be setup in such a way, that we can get rid of the astigmatism up to a 4th order term.

[2] Körner, et. al: Compact Aberration-Free Relay-Imaging Multi-Pass Layouts for High-Energy Laser Amplifiers

6.2 Exercise 14 - Q-Switch

You have a laser oscillator according to the sketch (roundtrip length is 3 m) below. The laser medium is Yb:CaF_2. The Pockels-cell (PC) is still switched offf and you adjust the output coupling under CW operation by turning the halve-wave plate and hence changing the transmission of the thin film polarizers (TFP).

a) *Where will your output beams be?*

b) *How can you find the point of minimum output coupling?*

c) *Starting at minimum output coupling the wavelength is 1050 nm. By increasing the output coupling the wavelength shifts to 1030 nm. Explain this effect.*

d) *What possibilities could you think of for adjusting the wavelength of your oscillator?*

You now want to operate the system in a cavity dumped mode:

e) *How shall the half-wave plate be aligned? Which phase shift got to be introduced by the Pockels-cell?*

f) *Describe how the Pockels-cells switching cycle go to be (temporal). How can you adjust the timing by monitoring the intra cavity intensity with a photodiode?*

g) *Which output pulse length and shape do you expect? (Assume an istantaneously switching Pockels-cell).*

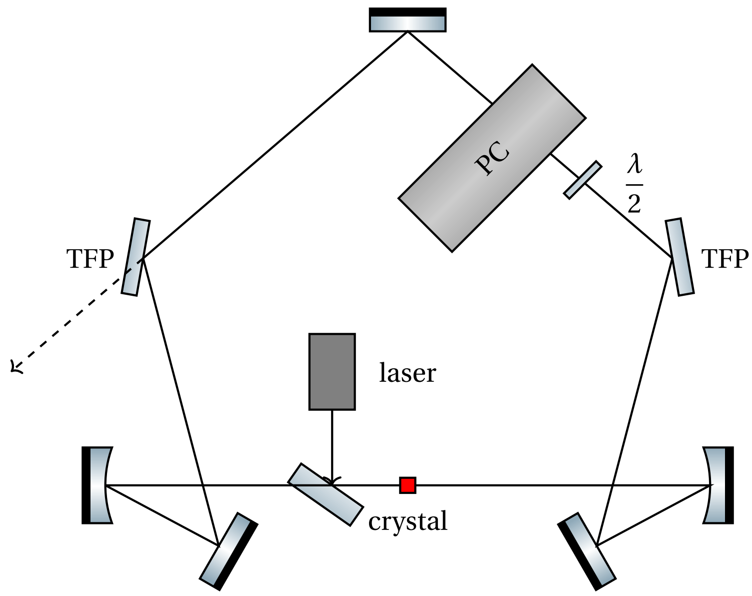


Fig. 13: Laser oscillator with a half wave plate and Pockels cell. The TFPs are used for output coupling.

a+b.) Solution:

The beam will be put out on the left thin film polarizer (TFP). If the laser generates CW radiation, that is s-polarized, the right TFP will be reflected. The half wave plate then induces a rotation of polarization by a certain angle (most probably 90°). The now p-polarized light will now be transmitted by the left TFP.

The point of minimum output coupling can be found by placing a photodiode behind the left TFP and measure the transmitted signal. Minimum output coupling can be achieved (Pockels cell on) when both TFP's are aligned, such as s-polarized light is completely reflected.

c+d.) Solution:

In order to achieve amplification we need an inversion $\beta > \beta_{\mathrm{eq}}$. The equilibrium inversion β_{eq}

$$\beta_{\mathrm{eq}} = \frac{\sigma_a}{\sigma_a + \sigma_e} = \frac{1}{1 + \exp\left(\frac{E_{ZL} - h\nu}{k_B T}\right)}. \tag{6.6}$$

is higher for smaller wavelengths (c. f. figure 5 on page 17). In the low output coupling regime there is a high intensity in the cavity thus reducing the inversion β. For low inversion only larger wavelengths may start to lase. If we increase the output coupling, the inversion rises as well, meaning that the more dominant emission line of 1030 nm now starts to lase.

We can furthermore adjust the output wavelength of our oscillator by using spectral shaping mirrors, which induces losses at certain wavelengths. This can be useful, e. g. for counteracting gain narrowing.

e+f.) Solution:

When working in cavity dumped mode, we want to trap the beam inside the cavity before opening it completely. In this case, we need to induce a $\lambda/2$ phase shift to counteract the effects of the half wave plate. For the cavity dumping process we simply switch off the Pockels cell. A laser light is then coupled out of the cavity within a single roundtrip. We can place a photodiode behind one of the (dielectric mirrors) to observe the intracavity intensity. When the intensity is saturated its time to open the cavity.

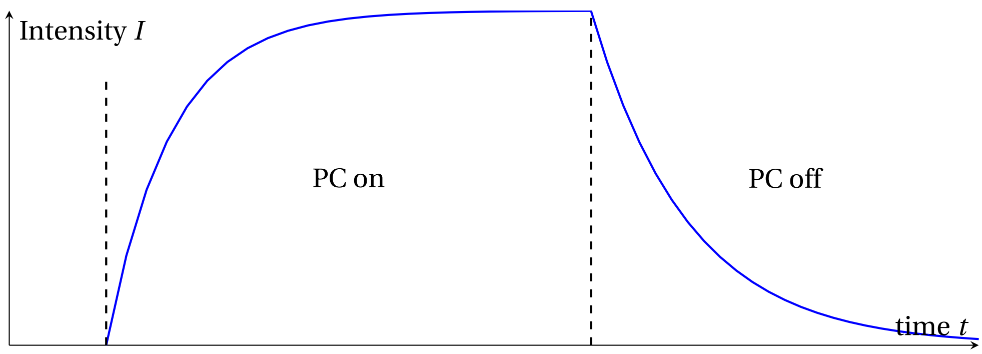


Fig. 14: Temporal evolution of the intracavity intensity in the oscillator.

g.) Solution:

For an instantaneously switching Pockels cell we expect a rectangular (temporal) pulse shape. The pulse length will correspond to the cavity length

$$\tau = \frac{3\,\mathrm{m}}{c} = 10\,\mathrm{ns}. \tag{6.7}$$