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High intensity relativistic optics

All Exercises

Winter term 2020/2021

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Contents

1 First Exercise

1.1 Light intensity

Calculate the light intensity I_L for a given vector potential $\boldsymbol{A}(x,t) = \hat{\boldsymbol{e}}_y A_0 \sin(k_L x - \omega_L t)$.

Solution: In general the electric field as a function of the scalar potential Φ and vector potential $\boldsymbol{A}$ is given as

$$\boldsymbol{E}_L = -\vec{\nabla}\Phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{1.1}$$

Since we assume no charge and current distribution, $\Phi = 0$ which results in

$$\boldsymbol{E}_L = -\frac{\partial \boldsymbol{A}}{\partial t} = \omega_L A_0 \cos(k_L x - \omega_L t)\hat{\boldsymbol{e}}_y. \tag{1.2}$$

The magnetic field is given as the curl of the vector potential and is therefore

$$\boldsymbol{B}_L = \vec{\nabla} \times \boldsymbol{A} = \begin{pmatrix} \partial x \\ \partial y \\ \partial z \end{pmatrix} \times \begin{pmatrix} 0 \\ A_0 \sin(k_L x - \omega t) \\ 0 \end{pmatrix} = A_0 k_L \cos(k_L x - \omega_L t)\hat{\boldsymbol{e}}_z. \tag{1.3}$$

We can introduce here the abbreviations $E_0 = \omega_L A_0$ and $B_0 = k_L A_0$, which yield the relation $c B_0 = E_0$, for which we assumed the dispersion relation $k_L = \omega_L / c$.

The intensity as defined as the time average of the magnitude of the poynting vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$. For no magnetization we can write $\mu_0 \boldsymbol{H} = \boldsymbol{B}$

$$\begin{aligned} I_L = \langle|\boldsymbol{S}|||\boldsymbol{S}|\rangle &= \langle|\boldsymbol{E}\times\boldsymbol{H}|||\boldsymbol{E}\times\boldsymbol{H}|\rangle = \frac{1}{\mu_0}\langle|\boldsymbol{E}\times\boldsymbol{B}|||\boldsymbol{E}\times\boldsymbol{B}|\rangle \\ |\boldsymbol{E}\times\boldsymbol{B}| &= E_0 B_0 \cos^2(k_L x - \omega_L t) \\ &= \frac{E_0 B_0}{\mu_0}\frac{1}{T}\int_0^T \cos^2(k_L x - \omega t)\,\mathrm{d}t, \end{aligned} \tag{1.4}$$

where $T = \frac{2\pi}{\omega}$ is the length of an optical cycle. We can easily solve this integral by using the addition formula $\cos^2(\varphi) = 0.5(1+\cos(2\varphi))$

$$\begin{aligned} I_L &= \frac{E_0 B_0}{2\mu_0}\frac{1}{T}\int_0^T [1+\cos(2k_L x - 2\omega t)]\,\mathrm{d}t \\ &= \frac{E_0 B_0}{2\mu_0}\frac{1}{T}\left[t - \frac{1}{2\omega}\sin(2k_L x - 2\omega t)\right|_0^T \\ &= \frac{E_0 B_0}{2\mu_0}\frac{1}{T}\Big(T - \frac{1}{2\omega}\underbrace{[\sin(2k_L x - 2\omega T) + \sin(2k_L x)]}_{=0}\Big) \\ &= \frac{E_0 B_0}{2\mu_0} = \frac{\varepsilon_0 c}{2}E_0^2. \end{aligned} \tag{1.5}$$

The sum of the two sines in the second to last line vanishes because a phase shift of $\pi = 2\omega T$ changes the sign of the sine.

1.2 Concentration of solar radiation

Assuming that all light power from the sun that is reaching the surface of the earth could be focussed to a small focal spot, how small would the focal spot area need to be in order to generate an intensity where an electron interacting with the associated electric field would least in the classical limit reach the speed of light? Assume that the total power emitted from the sun into the total solid angle is $3{,}86 \cdot 10^{26}\,\mathrm{W}$. *Which "practical" problems would have to be solved first to realize such a setup? Would it –at least in principle–be possible to use a sufficiently large lens?*

Solution: In the classical description an electron would reach the speed of light, when the maximum electron velocity divided by c is equal to one:

$$a_0 := \frac{eE_0}{\omega_L m_e c} = 1, \tag{1.6}$$

which can be identified as the dimensionless amplitude of the vector potential. This equation can be solved for the electric field amplitude

$$E_0 = \omega_L \frac{m_e c}{e} = \frac{2\pi}{\lambda} \frac{m_e c^2}{e}. \tag{1.7}$$

The intensity can be calculated by using (1.5)

$$I_L = \frac{\varepsilon_0 c}{2} E_0^2 = \frac{\varepsilon_0 c}{2} \left(\frac{2\pi}{\lambda} \frac{m_e c^2}{e} \right)^2 = \frac{\mu\mathrm{m}^2}{\lambda^2} 1{,}3682 \cdot 10^{18}\,\frac{\mathrm{W}}{\mathrm{cm}^2}. \tag{1.8}$$

For the mean wavelength of the sun $\lambda = 0{,}5\,\mu\mathrm{m}$ this leads to an intensity of

$$I_L = 5{,}473 \cdot 10^{18}\,\frac{\mathrm{W}}{\mathrm{cm}^2}. \tag{1.9}$$

If we want to calculate the whole incident power on the surface of the earth we need to multiply the intensity of the sunlight by the projected area of the earth towards the sun. This can be done as follows:

$$I_{\mathrm{earth}} = \frac{P_{\mathrm{sun}}}{4\pi R^2} \quad \Rightarrow P_L = P_{\mathrm{sun}} \frac{\pi r_{\mathrm{earth}}^2}{4\pi R^2} = 1{,}754 \cdot 10^{17}\,\mathrm{W}, \tag{1.10}$$

where $R = 1\,\mathrm{au}$ denotes the distance between the earth and the sun. The area of the focal spot can therefore be calculated as

$$A = \frac{P_L}{I_L} = \frac{1{,}754 \cdot 10^{17}\,\mathrm{W}}{1{,}3682 \cdot 10^{18}\,\frac{\mathrm{W}}{\mathrm{cm}^2}} = 0{,}032\,\mathrm{cm}^2. \tag{1.11}$$

If we want to focus the sunlight incident on the whole earth, we would need an optical system, that can collect the whole sunlight incident on the earth and focus it into a single spot. However, we can not construct such an optical system because, if we focus so much energy in this tiny spot, the receiver would heat up in such a way, that is surpasses the temperature of the sun. This would violate the second law of thermodynamics. Therefore there are certain limitations for a lens. For example we cannot construct lenses with a f-number $N = f/D$ less than 0.5. This means that there must be an upper limit for the concentration of solar radiation.

So even if we could build a suffciently large lens to collect the whole incident radiation, we can not construct it in such a way, that it will focus the whole energy into a spot of the size of $0{,}032\,\mathrm{cm}^2$, because this would violate the second law of thermodynamics.

1.3 Intensity measurement:

How can the intensity be measured in an experiment? Conceive of at least two methods to determine the intensity in an experiment dealing with relativistic effects. Keep in mind the huge value of the intensity and think of special physical effects that are directly related to the intensity.

Solution 1: As derived in the lecture, if we describe the movement of an Electron in a plane electromagnetic wave, the electron will be accelerated in the direction of field propagation. The solution of the equation of motion of the electron shows that it travels with an averaged drift velocity v_{drift} along the propagation axis. The drift velocity is given as

$$v_{\mathrm{drift}} = \left\langle \frac{x}{t} \middle| \frac{x}{t} \right\rangle_T = c\frac{a_0^2}{a_0^2+4} \quad \Rightarrow a_0^2 = \frac{4v_{\mathrm{drift}}}{c - v_{\mathrm{drift}}}. \tag{1.12}$$

According to equation (1.6) the normalized vector potential is connected to the electric field amplitude E_0. Therefore the electric field amplitude and thus the intensity can be written as

$$\begin{aligned} E_0^2 &= \left(\frac{\omega_L m_e c}{e}\right)^2 \frac{4v_{\mathrm{drift}}}{c - v_{\mathrm{drift}}} \\ \Rightarrow I_L &= \frac{\varepsilon_0 c}{2} E_0^2 = \frac{\varepsilon_0 c}{2}\left(\frac{\omega_L m_e c}{e}\right)^2 \frac{4v_{\mathrm{drift}}}{c - v_{\mathrm{drift}}}. \end{aligned} \tag{1.13}$$

By measuring the drift velocity of an accelerated electron, we can determine the intensity of the electromagnetic field.

Solution 2: Another approach to determine the intensity is to retrieve it indirectly by measuring the pulse energy, the focal area and the pulse duration, since the intensity is defined as:

$$I_L = \frac{\text{pulse energy}}{\text{pulse duration} \cdot \text{focal area}}. \tag{1.14}$$

The pulse energy can be measured by using an energy detector which is heated up, when a certain amount of energy is absorbed. The induced temperature difference can be calibrated to an energy quantity. If the repetition rate of the oscillator is known, we can then calculate the pulse energy.

The pulse duration can be measured by using an auto-correlator which uses a non-linear crystal producing the second harmonic of the laser frequency. The temporal information is translated into a spatially varying signal, from which shape the pulse duration can be deduced.

The focal area can be measured with a CCD-camera by using a lens of appropriate magnification. Then we can calculate the intensity of the pulse by using equation (1.14).

Solution 3: An alternative method is for example nonlinear thomson scattering, where a relativistic electron is accelerated by an intense laser pulse and is scattered elastically. By measuring the emission angle θ of the electron we can calculate the intensity as

$$I_0 = 0.28\left[\frac{\omega_0 m_e \gamma \sin(\theta_m)}{1+\cos(\theta_m)}\right] \cdot 10^{20}\,\frac{\mathrm{W}}{\mathrm{cm}^2}. \tag{1.15}$$

2 Second Exercise

2.1 Pressure by the ponderomotive force

Focusing a fraction of the JETI laser pulse containing a power of a 1 TW *to a spot of* 50 μm *in diameter onto a solid surface, suffices to create a plasma. The plasma will also be heated and consequently tries to expand. The ponderomotive force of the beam mainly acting on the region of critical density pushes the plasma back and causes a modification of the plasma density profile-abrupt changes of the density value corresponding to a spatial shift of the critical density value become observable.*

a) How much pressure is exerted by the ponderomotive force and to which mass does this correspond?

b) How large a density jump can be supported by the light pressure when the ion and electron temperatures are assumed to be $k_B T_i = k_B T_e = 1\,\mathrm{keV}$.

a.) Solution: We assume a Gaussian transversal beam profile, where the intensity drops to $1/e^2$ at $d/2 = w_0 = 50\,\mu\mathrm{m}$. We can write the profile as

$$I(r) = I_0 \exp\left(-2\left(\frac{r}{w_0}\right)^2\right), \tag{2.1}$$

where r is the transversal coordinate (because this problem is cylindrically symmetric). The total beam power can be calculated by integrating over the whole y-z-plane (propagation direction in x)

$$\begin{aligned} P &= \int_0^{2\pi}\int_0^{\infty} I_0 r \exp\left(-2\left(\frac{r}{w_0}\right)^2\right) \mathrm{d}r\,\mathrm{d}\varphi, \qquad u = r^2, \quad \mathrm{d}u = 2r\,\mathrm{d}r \\ &= \pi I_0 \int_0^{\infty} \exp\left(-\frac{2}{w_0^2}u\right)\mathrm{d}u = \frac{\pi}{2} w_0^2 I_0, \quad I_0 = \frac{2P}{\pi w_0^2}. \end{aligned} \tag{2.2}$$

Now we can use the result from the first exercise $I = \varepsilon_0 c/2 |E|^2$ to find an expression for the electric field in transversal direction:

$$E^2(r) = \frac{2}{\varepsilon_0 c}\frac{2P}{\pi w_0^2}\exp\left(-2\left(\frac{r}{w_0}\right)^2\right). \tag{2.3}$$

In the lecture we derived the formula for the *ponderomotive force*

$$\boldsymbol{F}_{\text{pond}} = -\frac{e^2}{4 m_e \omega_L^2}\vec{\nabla}(\boldsymbol{E}^2). \tag{2.4}$$

In this coordinate system the gradient reduces to the derivative with respect to r, so that we can write

$$\boldsymbol{F}_{\text{pond}} = -\frac{e^2}{\cancel{4} m_e \omega_L^2}\frac{\cancel{2}}{\varepsilon_0 c}\frac{\cancel{2}P}{\pi w_0^2}\frac{\mathrm{d}}{\mathrm{d}r}\exp\left(-2\left(\frac{r}{w_0}\right)^2\right)\hat{\boldsymbol{e}}_r. \tag{2.5}$$

Now we can resubstitute the angular frequency $\omega_L = 2\pi c/\lambda$ and perform the derivative

$$\boldsymbol{F}_{\text{pond}} = \frac{e^2\lambda^2 r}{\pi^3 m_e \varepsilon_0 c^3}\frac{P}{w_0^4}\exp\left(-2\left(\frac{r}{w_0}\right)^2\right)\hat{\boldsymbol{e}}_r. \tag{2.6}$$

We can now insert all the known quantities into equation (2.6) and use the central wavelength of the JETI laser system at $\lambda = 800\,\text{nm}$ (Ti:Sa Laser)

$$\boldsymbol{F}_{\text{pond}} = \underbrace{\frac{e^2(800\,\text{nm})^2}{\pi^3 m_e \varepsilon_0 c^3}\frac{1\,\text{TW}}{(25\,\mu\text{m})^4}}_{=\xi = 6{,}241\cdot 10^{-6}\,\frac{\text{N}}{\text{m}}} r\exp\left(-2\left(\frac{r}{w_0}\right)^2\right)\hat{\boldsymbol{e}}_r. \tag{2.7}$$

For $r = 12{,}5\,\mu\text{m}$ we get a ponderomotive force of

$$|\boldsymbol{F}_{\text{pond}}| = 47{,}4\,\text{pN}, \tag{2.8}$$

which corresponds to an acceleration of the electron of $a = 5{,}2\cdot 10^{19}\,\frac{\text{m}}{\text{s}^2}$. We can now obtain an expression for the pressure if we multiply the result for the ponderomotive force to the electron area density σ at the position r inside the beam.

b.) Solution: In order to calculate the density jump that can be supported by the light pressure we assume the position in the beam, where the ponderomotive force is maximal

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{F}_{\text{pond}}}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\xi r\exp\left(-2\left(\frac{r}{w_0}\right)^2\right)\right) = \xi\left(1-4\frac{r^2}{w_0^2}\right)\exp\left(-2\left(\frac{r}{w_0}\right)^2\right) \overset{!}{=} 0.\\ \Rightarrow r &= \frac{w_0}{2}.\end{aligned} \tag{2.9}$$

For that we have already calculated the ponderomotive force in (2.8). Now we demand an equilibrium between the temperature pressure of the plasma outside the laser beam and the temperature pressure with additional pressure exerted by the ponderomotive force inside the plasma. The condition for that is

$$F_{\text{pond}}\sigma + n_{\text{in}}k_B T = n_{\text{out}}k_B T, \tag{2.10}$$

where $n_{\text{in}}/n_{\text{out}}$ denotes the electron density inside/outside of the laser beam. Now we want to connect the electron area density at $r = w_0/2$ to the electron density n_{in}. For that consider an infinitesimal hollow cylinder of length l with its symmetry axis in the laser direction. We can write the number of electrons in the cylinder as

$$\begin{aligned} N = \sigma\cdot A_{\text{cyl}} \quad &\text{or} \quad N = n_{\text{in}}\cdot V_{\text{cyl}}\\ \Rightarrow \sigma = \frac{V_{\text{cyl}}}{A_{\text{cyl}}}n_{\text{in}} &= \frac{\pi r^2 l}{2\pi r l}n_{\text{in}} = \frac{r}{2}n_{\text{in}} = \frac{w_0}{4}n_{\text{in}}.\end{aligned} \tag{2.11}$$

We can put this into (2.10) which leads to

$$\begin{aligned} F_{\text{pond}}\frac{w_0}{4}n_{\text{in}} + n_{\text{in}}k_B T &= n_{\text{out}}k_B T\\ \Rightarrow \frac{n_{\text{out}}}{n_{\text{in}}} = 1 + \frac{F_{\text{pond}}w_0}{4k_B T} &= 1 + \frac{47{,}4\,\text{pN}\cdot 12{,}5\,\mu\text{m}}{4\,\text{keV}} = 1.92.\end{aligned} \tag{2.12}$$

The ratio between the outer and inner electron density is approximately 2:1.

2.2 Self focusing

Self-focusing of a cylindrically symmetric laser pulse of frequency ω_L occurs when it propagates through underdense plasma ($\omega_L > \omega_p$). In steady state, the beam's intensity profile and the density depression by the beam due to the ponderomotive force are related by a force balance. Prove the relation

$$n = n_0 \exp\left(-\frac{\varepsilon_0 \langle E^2 \rangle}{2 n_c k_B T}\right) \equiv n_0 \mathrm{e}^{-\alpha(r)}, \tag{2.13}$$

neglecting plasma heating ($k_B T = const.$) with $n_c = \omega_L^2 \varepsilon_0 m_e / e^2$ being the critical density of the plasma. The quantity $\alpha(0)$ is a measure of the relative importance of the ponderomotive pressure to the thermal pressure of the plasma.

Solution: We can prove the relation by assuming that the distribution function for electrons is Maxwellian having the following form:

$$f_e(v_e)\,\mathrm{d}v = A \cdot \exp\left(-\frac{1}{2}\frac{m_e v^2}{k_B T_e}\right)\mathrm{d}v\,. \tag{2.14}$$

Here, $f_e(v_e)\,\mathrm{d}v$ describes the number of electrons per unit volume $\mathrm{d}V$ with velocities v.

In the lecture we also showed that the averaged kinetic energy of the electron is equal to the ponderomotive potential

$$\overline{E_{\text{kin}}} = \frac{1}{2} m_e \langle v_e^2 \rangle = \frac{1}{4}\frac{e^2}{m_e \omega_L^2} E_0^2 =: \varphi_{\text{pond}}. \tag{2.15}$$

Now we boldly assume that by inserting the averaged velocity square $\langle v^2 \rangle$ into the Maxwellian distribution we obtain a value for the averaged electron density

$$n_e = n_0 \exp\left(-\frac{1}{2}\frac{m_e \langle v_e^2 \rangle}{k_B T_e}\right), \tag{2.16}$$

where n_0 is the electron density for resting electrons. We can now use relation (2.15) and insert it into (2.16). We get the electron density as a function of the ponderomotive potential

$$n_e = n_0 \exp\left(-\frac{\varphi_{\text{pond}}}{k_B T}\right) = n_0 \exp\left(-\frac{e^2 E^2}{4 m_e \omega_L^2 k_B T}\right). \tag{2.17}$$

We can now insert the time averaged electric field, which was already calculated in exercise one of the first sheet. We obtain $\langle E^2 \rangle = \frac{E^2}{2}$

$$n_e = n_0 \exp\left(-\frac{e^2 \langle E^2 \rangle}{2 m_e \omega_L^2 k_B T}\right) = n_0 \exp\left(-\frac{\varepsilon_0 e^2 \langle E^2 \rangle}{2 m_e \omega_L^2 \varepsilon_0 k_B T}\right), \tag{2.18}$$

where we can identify the blue terms as n_c which results in

$$n_e = n_0 \exp\left(-\frac{\varepsilon_0 \langle E^2 \rangle}{2 n_c k_B T}\right). \tag{2.19}$$

3 Third Exercise

3.1 Debye shielding

In a strictly steady-state situation, both the ions and the electrons obey a BOLTZMANN *distribution*

$$n_j = n_0 \exp\left(-\frac{q_j \phi}{k_B T_j}\right), \tag{3.1}$$

where $j \in \{e, i\}$. For the case of an infinite, transparent grid charged to a potential ϕ, show that the shielding distance is then given approximately by

$$\frac{1}{\lambda_D^2} = \frac{n_0 e^2}{\varepsilon_0}\left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i}\right). \tag{3.2}$$

Show that λ_D is determined by the temperature of the colder species.

Solution: In order to find an expression for the DEBYE length, we want to find the potential $\phi(x)$. We start by using the POISSON equation

$$\varepsilon_0 \Delta\phi = \varepsilon_0 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) = -\varrho(x), \tag{3.3}$$

where $\varrho(x)$ is the charge density. We can write $\varrho(x)$ as $-e(n_e(x) - Z\, n_i(x))$ with the electron density $n_e(x)$ and ion density $n_i(x)$. Now we use the BOLTZMANN distribution (3.1) and assume $Z = 1$ (for the case of hydrogen) leading to

$$\varepsilon_0 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) = e(n_e(x) - Z\, n_i(x)) = e\, n_0\left[\exp\left(\frac{e\phi}{k_B T_j}\right) - \exp\left(-\frac{e\phi}{k_B T_j}\right)\right] \tag{3.4}$$

since $q_e = -e$ and $q_i = +e$. If we now assume that thermal effects $k_B T_i$ dominate the potential energy, meaning $e\phi(x) \ll k_B T_i$ we can expand the exponential function using $\exp(x) \approx 1 + x$

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) &= \frac{e\, n_0}{\varepsilon_0}\left[(\cancel{1} + \frac{e\phi}{k_B T_j}) - (\cancel{1} - \frac{e\phi}{k_B T_j})\right]\\ &= \frac{e^2\, n_0}{\varepsilon_0}\left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i}\right)\phi.\end{aligned} \tag{3.5}$$

We can easily solve this differential equation which leads to the general solution

$$\phi(x) = \phi_0 \exp\left(+\sqrt{\frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i}\right)}x\right) + \phi_0' \exp\left(-\sqrt{\frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i}\right)}x\right). \tag{3.6}$$

We note that the potential does not diverge for $x \to \pm\infty$. For $x < 0$ we take the first part, whereas for $x > 0$ we take the second part of the sum (3.6). We can rewrite this as

$$\phi(x) = \phi_0 \exp\left(-\sqrt{\frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i}\right)}|x|\right). \tag{3.7}$$

The DEBYE length is defined as the distance, for which the externally applied electric potential in the plasma is reduced to $1/e$ of its peak value. Therefore we find

$$\phi(x) = \phi_0 \exp\left(-\frac{|x|}{\lambda_D}\right), \quad \Rightarrow \quad \frac{1}{\lambda_D^2} = \frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{k_B T_e} + \frac{1}{k_B T_i}\right). \tag{3.8}$$

If we now look at the expression for the DEBYE length (3.2), we see that it depends *inversely* on both temperatures of the electrons and the ions. However, the species with the lower temperature, has a bigger contribution to the expression (3.2) and will dominate the other term. Therefore the species with the lower temperature value will determine λ_D.

3.2 capacitor with plasma

The plates of a rectangular capacitor are at a potential difference of $\pm\phi_0$ and at a distance of $2R$. The capacitor is filled with a plasma, which has a Debye length of λ_D. What is the potential and the electric fields between the capacitor plates?
Discuss the limits of $R \gg \lambda_D$ and $R \ll \lambda_D$.

Solution: Since it is not specified in the task we assume that the capacitor plates extend infinitely in the y-z-plane and are positioned at $\pm R$ with distance $2R$. For $x < -R$ and $x > R$ we assume solid metal, which leads to the following boundary conditions for our problem

$$\phi(x) = \begin{cases} -\phi_0 & x \leq -R \\ +\phi_0 & x \geq +R \end{cases}. \tag{3.9}$$

A capacitor plate at $x = -R$ is not inherently different to a charged mesh of plasma with a potential ϕ. Therefore we assume that the capacitor plates just act as two meshes at $x = \pm R$. For a single mesh positioned at $x = 0$ we now how the potential for a given DEBYE length looks like:

$$\phi(x) = \phi_0 \exp\left(-\frac{|x|}{\lambda_D}\right). \tag{3.10}$$

We can now superposition the potentials for both capacitor plates and shift them to $x = \pm R$. This leads to

$$\phi(x) = -\tilde{\phi}\exp\left(-\frac{|x+R|}{\lambda_D}\right) + \tilde{\phi}\exp\left(-\frac{|x-R|}{\lambda_D}\right) \qquad \text{for } -R < x < R. \tag{3.11}$$

We note that $|x+R| = x+R$ and $|x-R| = R-x$.
By demanding $\phi(R) \equiv \phi_0$ we find an expression for $\tilde{\phi}$

$$\begin{aligned} \phi(R) &= -\tilde{\phi}\exp\left(-\frac{2R}{\lambda_D}\right) + \tilde{\phi} = \phi_0 \\ \Rightarrow \tilde{\phi} &= \frac{\phi_0}{1 - \exp\left(-\frac{2R}{\lambda_D}\right)}. \end{aligned} \tag{3.12}$$

We can summarize that into

$$\phi(x) = -\tilde{\phi}\exp\left(-\frac{x+R}{\lambda_D}\right) + \tilde{\phi}\exp\left(-\frac{R-x}{\lambda_D}\right) \qquad \text{for } \tilde{\phi} = \frac{\phi_0}{1-\exp\left(-\frac{2R}{\lambda_D}\right)}. \tag{3.13}$$

We can now calculate the electric field by using $\boldsymbol{E} = -\vec{\boldsymbol{\nabla}}\phi(x)$ which corresponds in 1D to $\boldsymbol{E} = -\frac{\partial}{\partial x}\phi(x)\hat{\boldsymbol{e}}_x$

$$\begin{aligned} \boldsymbol{E}(x) &= -\frac{\tilde{\phi}}{\lambda_D}\exp\left(-\frac{x+R}{\lambda_D}\right) - \frac{\tilde{\phi}}{\lambda_D}\exp\left(-\frac{R-x}{\lambda_D}\right)\hat{\boldsymbol{e}}_x \\ &= -\frac{\tilde{\phi}}{\lambda_D}\left(\exp\left(-\frac{x}{\lambda_D}\right) + \exp\left(\frac{x}{\lambda_D}\right)\right)\exp\left(-\frac{R}{\lambda_D}\right)\hat{\boldsymbol{e}}_x \\ &= -\frac{2\tilde{\phi}}{\lambda_D}\cosh\left(\frac{x}{\lambda_D}\right)\exp\left(-\frac{R}{\lambda_D}\right)\hat{\boldsymbol{e}}_x. \end{aligned} \tag{3.14}$$

Limit discussion We can now discuss the limits $R \gg \lambda_D$. For very big R we see that the exponential function in (3.13) falls very quickly when you move away from the plates and is zero everywhere else. The electric field is also zero everywhere, because $\exp\left(-\frac{R}{\lambda_D}\right)$ is small:

$$\phi(x) \to 0 \quad \text{and} \quad \boldsymbol{E}(x) \to 0. \tag{3.15}$$

This case corresponds to a material with a very small DEBYE length, i. e. a metal where the free charges immediately neutralize the potential at the capacitor plates.

The second limit is $R \ll \lambda_D$. Here we have small arguments in the exponential function, hence we can approximate $\exp(x) \approx 1 + x$:

$$\begin{aligned} \phi(x) &= -\tilde{\phi}\left(1 - \frac{x+R}{\lambda_D}\right) + \tilde{\phi}\left(1 - \frac{R-x}{\lambda_D}\right) \\ &= \tilde{\phi}\left(\frac{x+R}{\lambda_D} - \frac{R-x}{\lambda_D}\right) = \frac{2\tilde{\phi}}{\lambda_D}x. \end{aligned} \tag{3.16}$$

Correspondingly for the electric field

$$\begin{aligned} \boldsymbol{E}(x) &= -\frac{2\tilde{\phi}}{\lambda_D}\underbrace{\cosh\left(\frac{x}{\lambda_D}\right)}_{=1+\mathcal{O}(x^2)}\exp\left(-\frac{R}{\lambda_D}\right)\hat{\boldsymbol{e}}_x \\ &= -\frac{2\tilde{\phi}}{\lambda_D}\exp\left(-\frac{R}{\lambda_D}\right)\hat{\boldsymbol{e}}_x = \text{const.} \end{aligned} \tag{3.17}$$

This case corresponds to the classical solution of a capacitor in a vacuum with a linear potential $\phi(x)$ and a constant electric field between both plates. Here, the DEBYE-length is too huge to shield the external field of the capacitor at all. The plasma acts like a vacuum.

4 Fourth Exercise

4.1 Plasma frequency with massive ions

Improve the derivation of the plasma frequency ω_p by taking into account the mass of the ions. Assume that the ions in the ion layer all move together.

Solution: Similarly to the electrons we need to solve the equation of motion and the continuity equation for the ions:

$$m_i\left[\frac{\partial v_i}{\partial t}+\left(v_i\frac{\partial v_i}{\partial x}\right)\right]=+eE \tag{4.1}$$

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}(n_i v_i)=0, \tag{4.2}$$

whereas Poisson's equation remains unchanged

$$\varepsilon_0\frac{\partial E}{\partial x}=e(n_i-n_e). \tag{4.3}$$

We also make a linearized ansatz for the density, electric field and velocity for both ions and electrons:

$$\begin{aligned} n_e &= n_0+n_1 & n_i &= n_0+\tilde{n}_1 & n_\alpha &= n_{\alpha,0}\,\mathrm{e}^{\mathrm{i}[kx-\omega t]}\\ v_e &= v_1 & n_i &= n_0+\tilde{n}_1 & v_\alpha &= v_{\alpha,0}\,\mathrm{e}^{\mathrm{i}[kx-\omega t]}\\ E_e &= E_1 & E_i &= E_1 & E_\alpha &= E_{\alpha,0}\,\mathrm{e}^{\mathrm{i}[kx-\omega t]}. \end{aligned} \tag{4.4}$$

Analogously to the lecture we find the linearized differential equations for (4.1) and (4.2) (by cancelling out the exponentials and neglecting higher orders) as

$$-\mathrm{i}\omega m_e\, v_{10}=-E_{10} \qquad -\mathrm{i}\omega m_i\,\tilde{v}_{10}=+E_{10} \tag{4.5}$$

$$-\mathrm{i}\omega n_{10}+\mathrm{i}k\,n_0\,v_{10}=0 \qquad -\mathrm{i}\omega\tilde{n}_{10}+\mathrm{i}k\,n_0\,\tilde{v}_{10}=0. \tag{4.6}$$

We can use both equations to find an expression for n_{10} and $\tilde{n}_{10}$

$$n_{10}\overset{(4.6)}{=}\frac{k}{w}n_0 v_{10}\overset{(4.5)}{=}-\mathrm{i}\frac{ke}{\omega^2 m_e}n_0 E_{10} \tag{4.7}$$

$$\tilde{n}_{10}\overset{(4.6)}{=}\frac{k}{w}n_0\tilde{v}_{10}\overset{(4.5)}{=}+\mathrm{i}\frac{ke}{\omega^2 m_i}n_0 E_{10}. \tag{4.8}$$

We can now insert the results into (4.3)

$$\begin{aligned} \mathrm{i}kE_1 &= \frac{e}{\varepsilon_0}(n_i-n_e)=\frac{e}{\varepsilon_0}[(\cancel{n_0}+\tilde{n}_1)-(\cancel{n_0}+n_1)]\\ \Rightarrow \cancel{\mathrm{i}}\cancel{k}\cancel{E_{10}} &= \frac{e}{\varepsilon_0}[\tilde{n}_{10}-n_{10}]\underset{(4.8)}{\overset{(4.7)}{=}}\frac{e}{\varepsilon_0}\cancel{\mathrm{i}}\frac{\cancel{k}e}{\omega^2}\cancel{E_{10}}n_0\left(\frac{1}{m_i}+\frac{1}{m_e}\right). \end{aligned} \tag{4.9}$$

Therefore we can conclude for the plasma frequency $\omega\to\omega_p$

$$\omega_p^2=\frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{m_i}+\frac{1}{m_e}\right) \quad\Rightarrow\quad \omega_p=\sqrt{\frac{e^2 n_0}{\varepsilon_0}\left(\frac{1}{m_i}+\frac{1}{m_e}\right)}. \tag{4.10}$$

4.2 Potential of electric charge and ion background

Calculate the potential $\phi(\boldsymbol{r})$ of a system comprising an electric charge q at the origin, an immobile, homogeneous ion background and an electron distribution treated as a fluid characterized by a certain temperature T_e, which is in thermal equilibrium with the generated electro static potential $\phi(\boldsymbol{r})$. Start with a discussion on the assumption of a fixed ion background. Which conditions validate this assumption?

Solution: In order to calculate the potential we again start by using POISSON's equation $\Delta\phi(r) = -\varrho(r)/\varepsilon_0$. First we need to write down an expression for the charge density $\varrho(r)$. We write the positive ion background as the constant density $\varrho_i = e\,n_i$. We want to justify this step by saying that the ions (even for $Z = 1$ the hydrogen ion is 1836 times heavier than the electron) are much heavier than the electrons and will not be moving as much as the electrons due to their higher inertia. This assumption of static ion background is further justified by the condition, that the ion temperature T_i is smaller than T_e in a normal plasma. The electron distribution can be written down analogously to the lecture and the last exercise sheet as

$$n_e(r) = n_{e0}\exp\left(\frac{e\phi(r)}{k_B T_e}\right). \tag{4.11}$$

The charge in the origin can be described by a Delta Distribution $\varrho_q = q\delta(r)$. We can combine all charge densities to obtain

$$\varepsilon_0\Delta\phi(r) = -q\delta(r) - e\left(n_i - n_{e0}\exp\left(\frac{e\phi(r)}{k_B T_e}\right)\right). \tag{4.12}$$

We note that this differential equation for $r > 0$ looks similar to the Poisson equation for the Debye-shielding with $\Delta\phi(r) = \frac{\phi(r)}{\lambda_D^2}$ with the solution

$$\phi(r) = \phi_0\exp\left(-\frac{r}{\lambda_D}\right). \tag{4.13}$$

Therefore we can rewrite (4.12) as

$$\Delta\phi(r) - \frac{\phi(r)}{\lambda_D^2} = -\frac{q}{\varepsilon_0}\delta(r). \tag{4.14}$$

The Delta part of the Poisson equation can be solved by using the known Green's function for the Laplacian operator

$$\Delta G(\boldsymbol{r}) = \delta(\boldsymbol{r}) \quad \text{with} \quad G(\boldsymbol{r}) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}|}. \tag{4.15}$$

We can combine both solutions of the differential equation into a single potential

$$\phi(r) = -\frac{q}{4\pi\varepsilon_0}\frac{1}{r}\exp\left(-\frac{r}{\lambda_D}\right). \tag{4.16}$$

4.3 Deflection of a pulse

Consider a laser pulse obliquely incident on an exponential plasma profile, i. e. its (one-dimensional) density distribution as a function of the distance from the target surface, x, can be described by $n_e(x) = n_{e,0}(x)\exp\left(-|x|/L_p\right)$, where the target extends from $x = 0$ to larger values of x. There is no dependence of the density on the transverse dimensions (1D-situation). L_p is called the plasma scale length. How does the deflection of the pulse look like? Will the laser pulse reach the critical density?

Solution: We start the discussion by assuming that the laser beam is incident under an angle θ on the target. We can now describe the trajectory of the laser beam inside the plasma by using the refractive index defined as

$$\eta = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \tag{4.17}$$

where we use the density n_e dependent plasma frequency

$$\omega_p^2(x) = \frac{n_e(x)e^2}{\varepsilon_0 m} \Rightarrow \eta = \sqrt{1 - \frac{e^2}{\varepsilon_0 m\omega^2} n_{e0}\exp\left(-\frac{|x|}{L}\right)}. \tag{4.18}$$

We can see that the refractive index decreases for larger plasma densities. We will therefore observe that the beam is diffracted away from the target to larger angles until it may eventually propagate parallel to the target surface. This is illustrated in figure 1. In order to estimate,

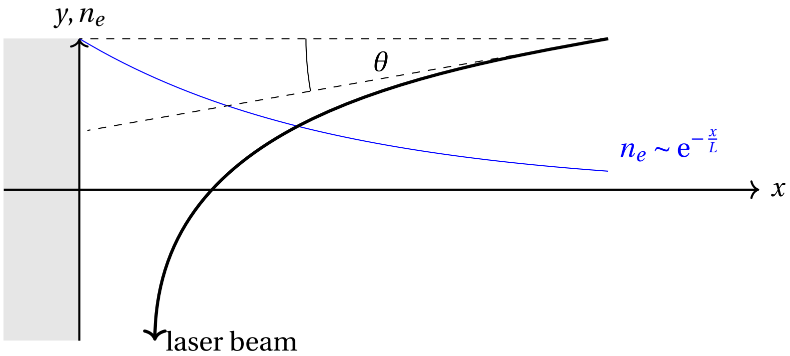

Fig. 1: Deflection of a laser beam with incident angle θ in a plasma.

whether or not the laser beam reaches the critical density of the plasma, we start by using Snell's law for diffraction which states

$$\eta(x)\sin\left(\phi(x)\right) = \text{const.} \tag{4.19}$$

The boundary condition for this is the incident angle θ at which our plasma is sufficiently underdense, so that we can estimate $\eta \approx 1$. Now we want to find the position x at which the laser beam will be deflected parallel to the target at $\phi = 90°$:

$$\sin\theta = \eta(x)\underbrace{\sin\left(\phi(x)\right)}_{=1}. \tag{4.20}$$

We now abbreviate the constants in (4.18) as α

$$\eta = \sqrt{1 - \alpha \exp\left(-\frac{x}{L}\right)} \overset{!}{=} \sin\theta \Rightarrow \exp\left(-\frac{x}{L}\right) = \frac{1 - \sin^2\theta}{\alpha}. \tag{4.21}$$

We note that the critical density n_{crit} of the plasma is given as

$$n_{\text{crit}} := \frac{\omega^2 \varepsilon_0 m}{e^2} \Rightarrow \alpha = \frac{n_{e0}}{n_{\text{crit}}}. \tag{4.22}$$

Now we can solve the equation (4.21) for x

$$\begin{aligned} x &= -L \ln\left(\frac{\cos^2\theta}{\alpha}\right) = -L \ln\left(\cos^2\theta \frac{n_{\text{crit}}}{n_{e0}}\right) \\ &= -L\left[\ln\left(\cos^2\theta\right) - \ln\left(\frac{n_{\text{crit}}}{n_{e0}}\right)\right]. \end{aligned} \tag{4.23}$$

Using $n_{\text{crit}} = n_{e0} \exp(-x_{\text{crit}}/L)$ we find

$$\begin{aligned} x &= -L\left[\ln\left(\cos^2\theta\right) - \frac{x_{\text{crit}}}{L}\right] \\ \Rightarrow x - x_{\text{crit}} &= -L \underbrace{\ln\left(\cos^2\theta\right)}_{<0}. \end{aligned} \tag{4.24}$$

We can see here that for reasonable angles $0 < \theta < \frac{\pi}{2}$ the difference $x - x_{\text{crit}}$ is greater than zero. Therefore we can conclude that the laser beam does not reach the depth x_{crit}, where the plasma is critical.

The only case, where $x = x_{\text{crit}}$ is achieved is, when $\ln(\cos\theta) = 0$, which is true for $\theta = 0$.

5 Fifth Exercise

5.1 Over the barrier ionization

When the laser intensity starts to get close to the so-called atomic intensity, which is the intensity associated with the atomic field strength as experienced by an electron on the first Bohr orbit in a hydrogen atom, the laser field becomes strong enough to distort the Coulomb field felt by the hydrogen atom's electron. Consider a modification of the Coulomb potential by a stationary, homogeneous electric field and determine the associated threshold intensity I of a laser pulse having this electric-field amplitude at which over-the-barrier ionization *(OTBI) sets in. Why is the required electric field strength of the laser smaller than the atomic field strength, i. e. the field which binds the electron to the nucleus?*

Solution: We assume that the Coulomb potential is modified by a homogeneous electric field $E_0 \cdot r$. We therefore obtain the following potential:

$$\Phi(r) = -\frac{Ze}{4\pi\varepsilon_0 r} - E_0 r. \tag{5.1}$$

The potential is depicted in figure 2. In order for OTBI to take place the potential must be

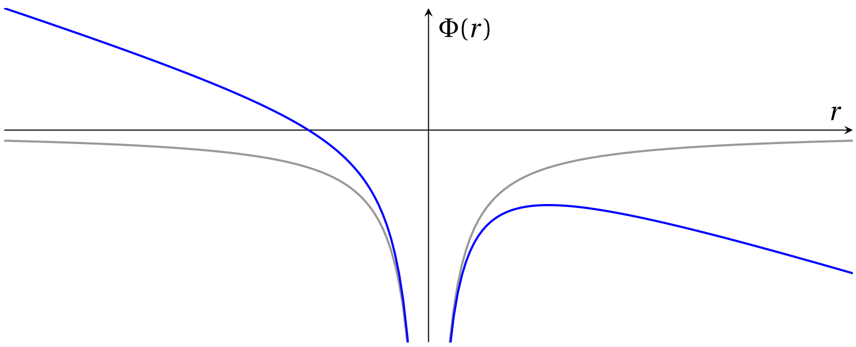

Fig. 2: Modified atomic potential (blue) through an external electric field vs. the undisturbed atomic potential (grey).

modified in such a way, that the electron of the hydrogen atom can escape the potential well. Therefore the maximum of the potential ($r > 0$) must be pushed to the ionization energy of hydrogen at 13,6 eV. We try to find the maximum of (5.1):

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} = \frac{Ze}{4\pi\varepsilon_0 r^2} - E_0 \stackrel{!}{=} 0. \tag{5.2}$$

We find the radius where the potential is at its maximum at

$$r_0^2 = \frac{Ze}{4\pi\varepsilon_0 E_0} \quad \Rightarrow \quad r_0 = \sqrt{\frac{Ze}{4\pi\varepsilon_0 E_0}} \approx 2\,\text{Å}. \tag{5.3}$$

We now demand, that the energy value $e{\cdot}\Phi(r_0)$ must be the ionization energy of the hydrogen ground state. This leads to

$$q\Phi(r_0) = -\sqrt{\frac{ZeE_0}{4\pi\varepsilon_0 E_0}} - \sqrt{\frac{ZeE_0}{4\pi\varepsilon_0 E_0}} = -\sqrt{\frac{ZeE_0}{\pi\varepsilon_0 E_0}} \overset{!}{=} -E_{\text{ion}}. \tag{5.4}$$

Solving this for the electric field amplitude E_0 yields

$$E_0 = \frac{(E_{\text{ion}})^2}{e^2}\frac{\pi\varepsilon_0}{Ze} \overset{Z=1}{=} 3{,}211\cdot 10^{10}\,\frac{\text{V}}{\text{m}}. \tag{5.5}$$

Now we can use the formula for the intensity that was derived in exercise sheet 1

$$\boxed{I = \frac{\varepsilon_0 c}{2}|E_0|^2 = \frac{c\cdot\varepsilon_0}{32}\frac{(4\pi\varepsilon_0)^2}{e^6 Z^2}(E_{\text{ion}})^4}. \tag{5.6}$$

Inserting the numerical value of (5.5) gives the result of

$$I_{\text{ion}} = 1{,}37\cdot 10^{14}\,\frac{\text{W}}{\text{cm}^2}. \tag{5.7}$$

If we compare the electric field amplitude to the field strength the nucleus exerts onto the electron we will find

$$E = \frac{1}{4\pi\varepsilon_0}\frac{e}{(0{,}5\,\text{Å})^2} = 5{,}76\cdot 10^{11}\,\frac{\text{V}}{\text{m}}. \tag{5.8}$$

We can see, that the electric field needed for *over-the-barrier ionization* is one order of magnitude smaller than the electric field which binds the electron to the nucleus. The reason is that the external electric field doesn‘t have to bend the potential all the way from zero to negative 13,6 eV, because the undisturbed potential at $r = r_0$ corresponds to an energy of

$$e\cdot\Phi_0(r_0) = \sqrt{\frac{ZeE_0}{4\pi\varepsilon_0 E_0}} = -6{,}8\,\text{eV}. \tag{5.9}$$

The electron is ionized before it has the energy 0 eV, therefore the external field can be smaller than the atomic field.

Another possible explanation is that the electron gains additional energy from the external field until it reaches $r = r_0$.

5.2 Self-focusing

A plasma with quasi-free electrons having a density of $n_{e,0}$ interacts in different ways with a high-power laser pulse propagating through it. Discuss qualitatively how ponderomotive self-focusing as well as relativistic self-focusing occur. Which thresholds might play a role and what are possible limitations to the different effects? How do ionization effects influence the propagation of the laser pulse?

Solution: The process of self-focusing of a laser pulse in a plasma can be due to ponderomotive forces or relativistic effects. We start by discussing the relativistic effects:

Due to the Gaussian shaped transversal intensity profile of the laser pulse, the plasma experiences different values of the normalized vector potential a_0. Due to

$$\langle \gamma \rangle = 1 + \frac{\langle a_0^2 \rangle}{2} \tag{5.10}$$

this leads to a spatial variation of the relativistic γ-factor of the electrons. Because the relativistic mass $m = m_{e,0} \cdot \gamma$ is proportional to γ, the mass of the electrons near the propagating axis will be higher. Considering the plasma frequency and the refractive index

$$\omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m}} \quad \text{and} \quad \eta = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{5.11}$$

a larger mass leads to a smaller plasma frequency and a higher refractive index η. The higher refractive index in the centre of the beam now acts like a convex lens which will focus the laser beam. The focusing is balanced by natural diffraction which increases for smaller beam diameters. In the lecture we derived relation describing the threshold, at which relativistic self-focussing occurs:

$$w_0^2 a_0^2 \geq 16 \frac{c^2}{\omega_p^2}. \tag{5.12}$$

For larger values $w_0^2 a_0^2$ relativistic self-focussing plays a role.

Now we discuss the effect of ponderomotive self focusing. We again assume a higher intensity on the propagation axis of the beam. This leads to a higher ponderomotive potential

$$\Phi_{\text{Pond}} := \frac{e^2}{4 m_e \omega_L^2} \boldsymbol{E}_s \propto I_L. \tag{5.13}$$

The higher ponderomotive forces pushes the electrons away towards smaller intensities which will decrease the density of electrons on the beam axis. Considering (5.11) this also leads to a smaller plasma frequency, thus increasing the refractive index. This causes the same focusing effect of the laser beam which is balanced by diffraction.

The pulse propagation is also influenced by ionization effects. The high intensity on the beam axis leads to more ionization processes which increase the electron density. Following the same arguments as before the increased density leads to a higher plasma frequency and a lower refractive index on the beam axis. This acts a concave lens causing defocusing effects. The beam gets defocused before it reaches the position of the vacuum focus. This effect is called ionization defocusing. This effect is limited by a depletion of ionized atoms. If all atoms are already ionized a higher laser intensity will not increase the defocusing effects.