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Fiber Optics

All Exercises

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Contents

1 Mirror waveguide

The slab mirror waveguide relies on a symmetrically embedded high refractive index film (dispersionless, refractive index n, film thickness d) sandwiched between two perfect metals.

1.1 Wave equation for TE-polarization

Derive the wave equation for the mirror waveguide in TE-polarization for the y-component (in Cartesian coordinates) of the electric field assuming:

- *a harmonic ansatz*
- *translational invariance along the y-direction*
- *propagation along the z-direction*
- *TE-condition: $E_x = E_z = H_y = 0$*

Solution:

Using the assumptions above we can formulate an ansatz for the electric field amplitude

$$\boldsymbol{E}(x,y,z) = E_y(x,z)\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t}\hat{\mathbf{e}}_y. \tag{1.1}$$

Now we can use Maxwells equations

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\partial}{\partial t}\mu_0 \boldsymbol{H} \quad \text{and} \quad \vec{\nabla} \times H = \frac{\partial}{\partial t}\boldsymbol{D} = \varepsilon_0 \varepsilon \frac{\partial}{\partial t}\boldsymbol{E} \tag{1.2}$$

to derive the wave equation. Applying the curl to Faradays law gives us

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \boldsymbol{E}\right) = \vec{\nabla}\underbrace{(\vec{\nabla}\cdot\boldsymbol{E})}_{=0} - \Delta \boldsymbol{E} &\overset{(1.2)}{=} -\mu_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \boldsymbol{H}\right) \\ &= -\underbrace{\mu_0\varepsilon_0}_{1/c_0^2}\varepsilon \frac{\partial^2}{\partial t^2}\boldsymbol{E}. \end{aligned} \tag{1.3}$$

Using the harmonic ansatz of the electric field we can substitute the time derivative with "$-\mathrm{i}\omega$" and find

$$\begin{aligned} \left(\Delta + \varepsilon\frac{\omega^2}{c_0^2}\right)\boldsymbol{E}(x,y,z) = 0 \\ \left(\Delta_t - \beta^2 + \varepsilon\frac{\omega^2}{c_0^2}\right)E_y(x,z) = 0. \end{aligned} \tag{1.4}$$

1.2 Wave equation for TM-polarization

Derive a similar wave equation in case of TM-polarization for the y-component of the magnetic field using the same assumption and the polarization conditions complementary to the TE-conditions.

Solution:

We now need to consider the magnetic field $\boldsymbol{H}$. Similar to equation (1.3) we take the curl on Amperes law

$$\begin{aligned}\vec{\nabla} \times \left(\vec{\nabla} \times \boldsymbol{H}\right) = \vec{\nabla}\underbrace{(\vec{\nabla} \cdot \boldsymbol{H})}_{=0} - \Delta \boldsymbol{H} &\overset{(1.2)}{=} \varepsilon_0 \varepsilon \frac{\partial}{\partial t}\left(\vec{\nabla} \times \boldsymbol{E}\right) \\ &= -\underbrace{\mu_0 \varepsilon_0}_{1/c_0^2} \varepsilon \frac{\partial^2}{\partial t^2} \boldsymbol{H}. \end{aligned} \tag{1.5}$$

Analogously to the task before we can than derive the wave equation as

$$\left(\Delta_t - \beta^2 + \varepsilon \frac{\omega^2}{c_0^2}\right) H_y(x, z) = 0. \tag{1.6}$$

1.3 Dispersion equation for TE-polarization

Use the following ansatz for the modal fields to find the dispersion equation in TE-polarization taking into account the boundary conditions

$$E_y = E_y^0 \cos(k_\perp x) \quad \textit{and} \quad E_y\left(x = \pm\frac{d}{2}\right) = 0. \tag{1.7}$$

What happens to the dispersion relation in case of the anti-symmetric mode?

Solution:

Substituting ansatz (1.7) into (1.4) yields

$$\left(\Delta_t - \beta^2 + \varepsilon\frac{\omega^2}{c_0^2}\right)E_y^0 \cos(k_\perp x) = \left(-k_\perp^2 - \beta^2 + \varepsilon\frac{\omega^2}{c_0^2}\right)E_y^0 \cos(k_\perp x) = 0. \tag{1.8}$$

In order to always fulfill this equation the following must hold:

$$k_\perp^2 + \beta^2 = \varepsilon\frac{\omega^2}{c_0^2} := k^2 \quad \text{with} \quad k = nk_0. \tag{1.9}$$

Using the boundary condition we have

$$\begin{aligned} \cos\left(\pm k_\perp \frac{d}{2}\right) = 0 \quad &\Rightarrow \quad k_\perp \frac{d}{2} = \frac{\pi}{2} + n\pi \quad \text{with} \quad n \in \mathbb{N}_0 \\ &\Rightarrow \quad k_\perp = \frac{2}{d}\pi\left(n + \frac{1}{2}\right) = \frac{\pi}{d}m \quad \text{with} \quad m = 1,3,5,\ldots \end{aligned} \tag{1.10}$$

Then we can solve (1.9) for β and find

$$\beta = \sqrt{k^2 - k_\perp^2} = \sqrt{n^2k_0^2 - \left(\frac{\pi}{d}m\right)^2} = nk_0\sqrt{1 - \left(\frac{m\pi}{dnk_0}\right)^2}. \tag{1.11}$$

If we now introduce an effective index $n_{\text{eff}} = \frac{\beta}{k_0}$ we find the disperion equation of a mirror waveguide

$$n_{\text{eff}} = n\sqrt{1 - \left(\frac{m\pi}{dnk_0}\right)^2} \quad \text{with} \quad m = 1,3,5,\ldots \tag{1.12}$$

For the anti-symmetric mode m will take even values $m = 2,4,6,\ldots$

1.4 Propagation constant

Plot the propagation constant as function of wavelength for the three lowest order modes ($n = 1.45, d = 2\,\mu\text{m}, 0.5 < \lambda < 6\,\mu\text{m}$). Derive general expressions for the cut-off wavelength (at which $\beta = 0$) and plot the cut-off wavelength for the mode order (combining solution of TE- and TM-polarization) $1 \leq m \leq 5$ for the above mentioned parameters.

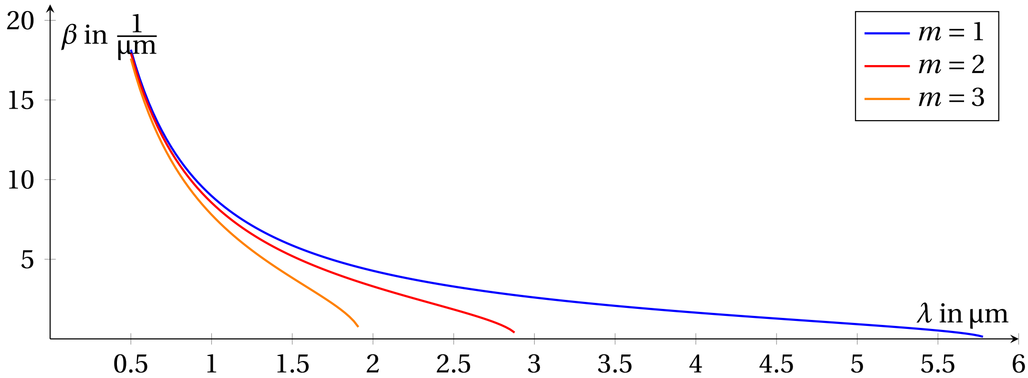

Fig. 1: Propagation constant as a function of wavelength for the three lowest order modes for a glass mirror guide.

The cut-off wavelength can be determined by setting (1.11) to zero:

$$0 = \sqrt{1 - \left(\frac{m\pi}{dnk_0}\right)^2} \quad \Rightarrow \quad 1 = \frac{m\lambda}{2dn} \quad \Rightarrow \quad \lambda_c = \frac{2dn}{m}. \tag{1.13}$$

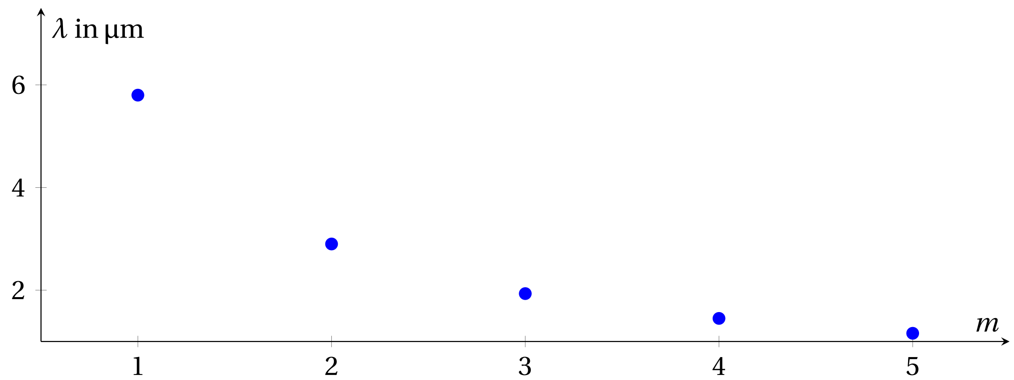

Fig. 2: Cut-off wavelength λ_c as a function of the mode order.

1.5 Group velocity dispersion

Derive analytic equations for the group velocity dispersion and show that causality holds. Plot the relative group velocity (v_g/c_0) as a function of wavelength for the configuration defined in Exercise 1.4. What happens close to the cut-off.

Solution:

The group velocity is given as

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta} \quad \Rightarrow \quad \frac{1}{v_g} = \frac{\mathrm{d}\beta}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{n}{c}\omega\sqrt{1-\left(\frac{m\pi c}{dn\omega}\right)^2}\right). \tag{1.14}$$

For a shorter notation we summarize the constants $\frac{m\pi c}{dn}$ into a new constant α. Then we can formally calculate the derivative[1]

$$\frac{\mathrm{d}\beta}{\mathrm{d}\omega} = \frac{n}{c}\left(\sqrt{1-\frac{\alpha^2}{\omega^2}} + \omega\frac{\frac{\alpha^2}{\omega^3}}{\sqrt{1-\frac{\alpha^2}{\omega^2}}}\right) = \frac{n}{c}\frac{1}{\sqrt{1-\frac{\alpha^2}{\omega^2}}}\left(1-\frac{\alpha^2}{\omega^2}+\frac{\alpha^2}{\omega^2}\right) = \frac{n}{c}\frac{1}{\sqrt{1-\frac{\alpha^2}{\omega^2}}}. \tag{1.15}$$

Thus the group velocity dispersion is

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta} = \left(\frac{n}{c}\frac{1}{\sqrt{1-\frac{\alpha^2}{\omega^2}}}\right)^{-1} = \frac{c}{n}\sqrt{1-\left(\frac{m\pi}{dnk_0}\right)^2}. \tag{1.16}$$

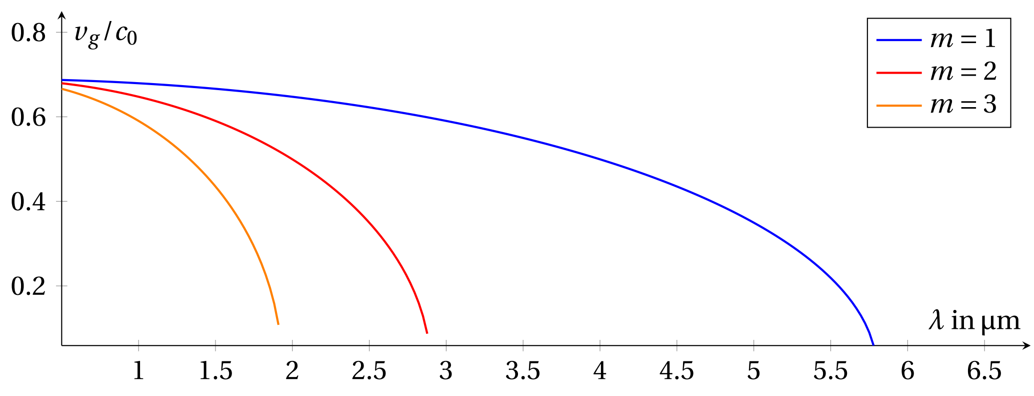

Fig. 3: Relative group velocity as a function of wavelength for the previously defined configuration. Close to the cut-off the relative velocity drops down to zero very quickly.

[1] It is much easier to put ω into the square root, then we don´t have to apply the product rule.

2 The symmetric planar slab waveguide

Given is a symmetric dielectric slab waveguide in TE polarization (see sketch below). The propagation direction is along the z-axis and the waveguide is invariant along the y-direction. The modes in this structure are given by:

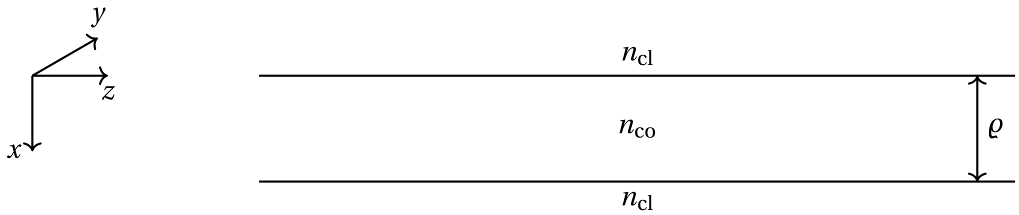

Fig. 4: Geometry of the planar slab waveguide

$$E_y = \begin{cases} A\mathrm{e}^{-\gamma_1(x-\varrho)} & x \geq \varrho \\ B\sin(\kappa x) + C\cos(\kappa x) & 0 \leq x < \varrho \\ D\mathrm{e}^{\gamma_3 x} & x < 0 \end{cases} \tag{2.1}$$

2.1 Derivation of γ_1, γ_3 and κ

Use the wave equation to derive expressions for γ_1, γ_3 and κ.

Solution:

We can formulate the wave equations as follows:

$$\left(\frac{\partial^2}{\partial x^2} + n_{\mathrm{co}}^2 k_0^2 - \beta^2\right) E(x) = 0 \tag{2.2}$$

$$\left(\frac{\partial^2}{\partial x^2} + n_{\mathrm{cl}}^2 k_0^2 - \beta^2\right) E(x) = 0. \tag{2.3}$$

Using the ansatz for the fields (2.1) we can find

$$\kappa^2 = n_{\mathrm{co}}^2 k_0^2 - \beta^2 \quad \text{and} \quad \gamma_3^2 = \gamma_1^2 = \beta^2 - n_{\mathrm{cl}}^2 k_0^2. \tag{2.4}$$

2.2 Boundary conditions

Write down the boundary conditions for this waveguide configuration.

Solution:

The boundary conditions are obtained by demanding continuity of the transverse electric fields and their derivatives at the two interfaces. At $x = 0$ we find

$$De^{\gamma_3 0} = B\sin(\kappa 0) + C\cos(\kappa 0) \quad \Rightarrow \quad D = C \tag{2.5}$$

$$D\gamma_3 e^{\gamma_3 0} = B\kappa\cos(\kappa 0) - C\kappa\sin(\kappa 0) \quad \Rightarrow \quad D\gamma_3 = B\kappa. \tag{2.6}$$

For the boundary at $x = \varrho$ we find

$$A = B\sin(\kappa\varrho) + C\cos(\kappa\varrho) \tag{2.7}$$

$$-A\gamma_1 = B\kappa\cos(\kappa\varrho) - C\kappa\sin(\kappa\varrho). \tag{2.8}$$

2.3 Dispersion relation

Derive the dispersion relation of the modes of this slab waveguide configuration. The final expression should have the form

$$\frac{2\kappa\gamma}{\kappa^2 - \gamma^2} = \tan(\kappa\varrho) \quad \textit{with} \quad \gamma = \gamma_1 = \gamma_3. \tag{2.9}$$

Solution:

We start by equating equations (2.7), (2.8) and using $C\gamma_3 = B\kappa$

$$\begin{aligned}
B\sin(\kappa\varrho) + C\cos(\kappa\varrho) &= \frac{C\kappa}{\gamma_1}\sin(\kappa\varrho) - \frac{B\kappa}{\gamma_1}\cos(\kappa\varrho) \\
\Rightarrow B\sin(\kappa\varrho) + \frac{\kappa}{\gamma_3}B\cos(\kappa\varrho) &= \frac{B\kappa^2}{\gamma_1\gamma_3}\sin(\kappa\varrho) - \frac{B\kappa}{\gamma_1}\cos(\kappa\varrho) \\
\Rightarrow \left(1 - \frac{\kappa^2}{\gamma_1\gamma_3}\right)\sin(\kappa\varrho) &= -\kappa\left(\frac{1}{\gamma_1} + \frac{1}{\gamma_3}\right)\cos(\kappa\varrho) \\
\Rightarrow \frac{\gamma_1\gamma_3 - \kappa^2}{\gamma_1\gamma_3}\sin(\kappa\varrho) &= -\kappa\left(\frac{\gamma_1 + \gamma_3}{\gamma_1\gamma_3}\right)\cos(\kappa\varrho) \\
\Rightarrow \tan(\kappa\varrho) &= \kappa\frac{\gamma_1 + \gamma_3}{\kappa^2 - \gamma_1\gamma_3} = \frac{2\kappa\gamma}{\kappa^2 - \gamma^2}.
\end{aligned} \tag{2.10}$$

2.4 Effective indices

Plot the right- and left-handed side of the dispersion equation as a function of n_{eff} for $\varrho = 5\,\mu\mathrm{m}$, $\lambda_0 = 1\,\mu\mathrm{m}$, $n_{co} = 1.5$, $n_{cl} = 1.45$ within the range $n_{cl} < n_{\mathrm{eff}} < n_{co}$. Remember that $n_{\mathrm{eff}} = \frac{\beta}{k_0}$. Find or read off the effective indices of the fundamental mode and the two next higher-order modes.

We can modify the relations (2.4) as a function of the effective index

$$\kappa^2 = k_0^2(n_{\mathrm{co}}^2 - n_{\mathrm{eff}}^2) \quad \text{and} \quad \gamma^2 = k_0^2(n_{\mathrm{eff}} - n_{\mathrm{cl}}^2). \tag{2.11}$$

Substituting this into (2.10) results in

$$\tan\left(k_0\varrho\sqrt{n_{\mathrm{co}}^2 - n_{\mathrm{eff}}^2}\right) = \frac{2\sqrt{(n_{\mathrm{co}}^2 - n_{\mathrm{eff}}^2)(n_{\mathrm{eff}}^2 - n_{\mathrm{cl}}^2)}}{(n_{\mathrm{co}}^2 - n_{\mathrm{eff}}^2) - (n_{\mathrm{eff}}^2 - n_{\mathrm{cl}}^2)} = \frac{2\sqrt{(n_{\mathrm{co}}^2 - n_{\mathrm{eff}}^2)(n_{\mathrm{eff}}^2 - n_{\mathrm{cl}}^2)}}{n_{\mathrm{co}}^2 - 2n_{\mathrm{eff}}^2 + n_{\mathrm{cl}}^2}. \tag{2.12}$$

We can now find the solutions in a graphical way as displayed in figure 5.

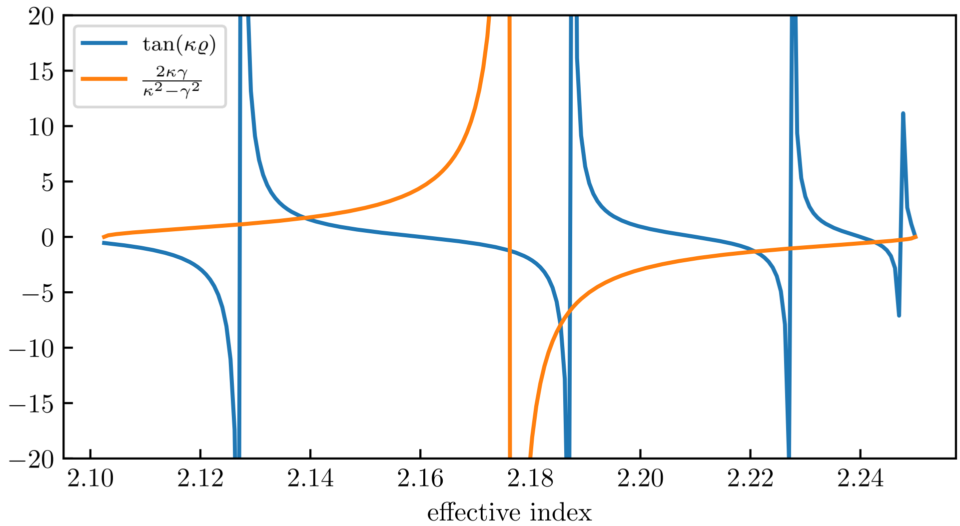

Fig. 5: Plot of the left hand side and right hand side of (2.12). The three solutions we can obtain are $n_1 = 2.139$ (fundamental mode), $n_3 = 2.220$ (first higher mode), $n_4 = 2.242$ (second higher mode).

2.5 Poynting vector

Calculate the transverse Poynting vector distribution of the three modes discussed in the previous task. Use $S_z = \frac{1}{2}\mathrm{Re}(\boldsymbol{E} \times \boldsymbol{H}^*)$.

Solution:

We can rewrite the Poynting vector using $|\boldsymbol{H}| = \frac{n_{\text{eff}}}{c_0\mu_0}|\boldsymbol{E}|$

$$S_z = \frac{1}{2}\text{Re}(\boldsymbol{E}\times\boldsymbol{H}^*) = \frac{n_{\text{eff}}}{c_0\mu_0}|\boldsymbol{E}|^2 = \frac{\varepsilon_0 n_{\text{eff}}}{2}|\boldsymbol{E}|^2. \tag{2.13}$$

3 Optical fiber

3.1 Transverse EM-components

Show that all transverse EM-components depend on the longitudinal components E_z and H_z. Use Maxwells equations

$$\vec{\nabla} \times \boldsymbol{H} = -\mathrm{i}\omega\varepsilon\varepsilon_0 \boldsymbol{E} \quad \textit{and} \quad \vec{\nabla} \times \boldsymbol{E} = \mathrm{i}\omega\mu_0 \boldsymbol{H} \tag{3.1}$$

in cylindrical coordinates.

Solution:

We start by explicitly writing down the curl operator in cylindrical coordinates

$$\left[\frac{1}{r}\frac{\partial H_z}{\partial\varphi} - \frac{\partial H_\varphi}{\partial z}\right]\hat{\mathbf{e}}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{\mathbf{e}}_\varphi + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rH_\varphi\right) - \frac{\partial H_r}{\partial\varphi}\right]\hat{\mathbf{e}}_z = -\mathrm{i}\omega\varepsilon\varepsilon_0\boldsymbol{E} \tag{3.2}$$

$$\left[\frac{1}{r}\frac{\partial E_z}{\partial\varphi} - \frac{\partial E_\varphi}{\partial z}\right]\hat{\mathbf{e}}_r + \left[\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right]\hat{\mathbf{e}}_\varphi + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rE_\varphi\right) - \frac{\partial E_r}{\partial\varphi}\right]\hat{\mathbf{e}}_z = \mathrm{i}\omega\mu_0\boldsymbol{H}. \tag{3.3}$$

Furthermore we want to assume fields of the form $E(x, y, z) = u(x, y)\mathrm{e}^{\mathrm{i}\beta z}$ with a phase factor $\beta = n_{\mathrm{eff}}k_0$. From that we can read of the transverse components of electric and magnetic field:

$$E_r = \frac{\mathrm{i}}{\omega\varepsilon\varepsilon_0}\left[\frac{1}{r}\frac{\partial H_z}{\partial\varphi} - \mathrm{i}\beta H_\varphi\right] \qquad E_\varphi = \frac{\mathrm{i}}{\omega\varepsilon\varepsilon_0}\left[\mathrm{i}\beta H_r - \frac{\partial H_z}{\partial r}\right] \tag{3.4}$$

$$H_r = \frac{1}{\mathrm{i}\omega\mu_0}\left[\frac{1}{r}\frac{\partial E_z}{\partial\varphi} - \mathrm{i}\beta E_\varphi\right] \qquad H_\varphi = \frac{1}{\mathrm{i}\omega\mu_0}\left[\mathrm{i}\beta E_r - \frac{\partial E_z}{\partial r}\right]. \tag{3.5}$$

For the radial component of the electric field (3.4) we substitute the φ component of the magnetic field (3.5) and find with $k_0 = \omega/c_0$

$$E_r = \frac{\mathrm{i}}{\omega\varepsilon\varepsilon_0}\left[\frac{1}{r}\frac{\partial H_z}{\partial\varphi} - \frac{\beta}{\omega\mu_0}\left[\mathrm{i}\beta E_r - \frac{\partial E_z}{\partial r}\right]\right]$$

$$\underbrace{\left(1 - \frac{\beta^2}{\omega^2\varepsilon\varepsilon_0\mu_0}\right)}_{1-\frac{n_{\mathrm{eff}}^2}{\varepsilon}} E_r = \frac{\mathrm{i}}{\omega\varepsilon\varepsilon_0}\left[\frac{1}{r}\frac{\partial H_z}{\partial\varphi} + \frac{\beta}{\omega\mu_0}\frac{\partial E_z}{\partial r}\right] \quad \text{with} \quad c_0^2 = \frac{1}{\mu_0\varepsilon_0}$$

$$E_r = \frac{\mathrm{i}c_0}{\omega\varepsilon}\frac{1}{1 - \frac{n_{\mathrm{eff}}^2}{\varepsilon}}\left[\frac{1}{\varepsilon_0 c_0 r}\frac{\partial H_z}{\partial\varphi} + \frac{\beta}{\omega c_0\mu_0\varepsilon_0}\frac{\partial E_z}{\partial r}\right]$$

$$E_r = \frac{\mathrm{i}}{k_0(\varepsilon - n_{\mathrm{eff}}^2)}\left[\frac{1}{\varepsilon_0 c_0 r}\frac{\partial H_z}{\partial\varphi} + n_{\mathrm{eff}}\frac{\partial E_z}{\partial r}\right]. \tag{3.6}$$

We can also find the φ compoenten of the magnetic field by substituting E_r instead

$$\begin{aligned}
H_\varphi &= \frac{1}{\mathrm{i}\omega\mu_0}\left[-\frac{\beta}{\omega\varepsilon\varepsilon_0}\left[\frac{1}{r}\frac{\partial H_z}{\partial\varphi} - \mathrm{i}\beta H_\varphi\right] - \frac{\partial E_z}{\partial r}\right] \\
\left(1-\frac{\beta^2}{\omega^2\mu_0\varepsilon_0\varepsilon}\right)H_\varphi &= \frac{1}{\mathrm{i}\omega\mu_0}\left[-\frac{\beta}{\omega\varepsilon\varepsilon_0}\frac{1}{r}\frac{\partial H_z}{\partial\varphi} - \frac{\partial E_z}{\partial r}\right] \\
\left(1-\frac{n_{\mathrm{eff}}^2}{\varepsilon}\right)H_\varphi &= \frac{\mathrm{i}c_0}{\omega\varepsilon}\left[-\frac{\beta}{\omega c_0\mu_0\varepsilon_0}\frac{1}{r}\frac{\partial H_z}{\partial\varphi} - \frac{\varepsilon}{c_0\mu_0}\frac{\partial E_z}{\partial r}\right] \\
H_\varphi &= \frac{-\mathrm{i}}{k_0(\varepsilon - n_{\mathrm{eff}}^2)}\left[\frac{n_{\mathrm{eff}}}{r}\frac{\partial H_z}{\partial\varphi} + \varepsilon\varepsilon_0 c_0\frac{\partial E_z}{\partial r}\right]. \qquad (3.7)
\end{aligned}$$

The other two components can be derived analogously and are:

$$\begin{aligned}
E_\varphi &= -K\left(-\frac{n_{\mathrm{eff}}}{r}\frac{\partial E_z}{\partial\varphi} + \frac{1}{\varepsilon_0 c_0}\frac{\partial H_z}{\partial r}\right) \\
H_r &= K\left(n_{\mathrm{eff}}\frac{\partial H_z}{\partial r} - \frac{\varepsilon\varepsilon_0 c_0}{r}\frac{\partial E_z}{\partial\varphi}\right) \quad \text{with} \quad K = \frac{\mathrm{i}}{k_0(\varepsilon - n_{\mathrm{eff}}^2)}. \qquad (3.8)
\end{aligned}$$

3.2 Application of boundary conditions

Use boundary condition BC3 ($A = A_E$) and BC4 ($A = A_H$) to find expressions for A. Use the relation

$$J'_m(x) = \frac{m}{x}J_m(x) - J_{m+1}(x) \quad \textit{and} \quad K'_m(x) = \frac{m}{x}K_m(x) - K_{m+1}(x). \qquad (3.9)$$

Solution:

We start by writing down the two boundary conditions

$$\text{BC3: } E_\varphi^{\mathrm{co}} = E_\varphi^{\mathrm{cl}} \qquad \text{BC4: } H_\varphi^{\mathrm{co}} = H_\varphi^{\mathrm{cl}} \qquad (3.10)$$

Furthermore we want to list again the ansatz functions for the z-component of the fields: Now we take the expressions from Task 1 and insert the ansatz vor the fields E_z and H_z. We

Table 1: Ansatz functions for the longitudinal fields. Note that $B = 1$ since only the ratio between A and B is relevant.

	E_z	H_z
core	$A\dfrac{J_m(UR)}{J_m(U)}\cos(m\varphi)$	$B\dfrac{J_m(UR)}{J_m(U)}\sin(m\varphi)$
cladding	$A\dfrac{K_m(WR)}{K_m(W)}\cos(m\varphi)$	$B\dfrac{K_m(WR)}{K_m(W)}\sin(m\varphi)$

start with BC3 ($r=\varrho, R=1$)

$$\begin{aligned}
E_\varphi^{\mathrm{co}} &= -K_{\mathrm{co}}\left(-\frac{n_{\mathrm{eff}}}{r}\frac{\partial E_z}{\partial\varphi}+\frac{1}{\varepsilon_0 c_0}\frac{\partial H_z}{\partial r}\right)\\
&= -K_{\mathrm{co}}\left(-\frac{n_{\mathrm{eff}}}{r}\frac{\partial}{\partial\varphi}\left[A_E\cos(m\varphi)\right]+\frac{1}{\varepsilon_0 c_0}\frac{\partial}{\partial r}\left[\frac{J_m(U)}{J_m(U)}\sin(m\varphi)\right]\right)\\
&= -K_{\mathrm{co}}\left(\frac{n_{\mathrm{eff}}}{r}mA_E\sin(m\varphi)+\frac{1}{\varepsilon_0 c_0\varrho}\sin(m\varphi)\left[m-\frac{J_{m+1}(U)}{J_m(U)}U\right]\right)\\
E_\varphi^{\mathrm{cl}} &\overset{!}{=} -K_{\mathrm{cl}}\left(\frac{n_{\mathrm{eff}}}{r}mA_E\sin(m\varphi)+\frac{1}{\varepsilon_0 c_0\varrho}\sin(m\varphi)\left[m-\frac{K_{m+1}(W)}{K_m(W)}W\right]\right).
\end{aligned} \tag{3.11}$$

We can solve this equation for A_E

$$\begin{aligned}
A_E &= \frac{\dfrac{\sin(m\varphi)}{\varepsilon_0 c_0\varrho}\left(-\dfrac{K_{m+1}(W)}{K_m(W)}W+\dfrac{J_{m+1}(U)}{J_m(U)}U\right)}{\dfrac{n_{\mathrm{eff}}}{\varrho}m\sin(m\varphi)(K_{\mathrm{cl}}-K_{\mathrm{co}})}\\
&= \frac{1}{\varepsilon_0 c_0 n_{\mathrm{eff}} m(K_{\mathrm{cl}}-K_{\mathrm{co}})}\left(K_{\mathrm{cl}}\frac{K_{m+1}(W)}{K_m(W)}W-K_{\mathrm{co}}\frac{J_{m+1}(U)}{J_m(U)}U\right).
\end{aligned} \tag{3.12}$$

We repeat all steps with BC4 ($r=\varrho, R=1$)

$$\begin{aligned}
H_\varphi^{\mathrm{co}} &= K_{\mathrm{co}}\left(\frac{n_{\mathrm{eff}}}{\varrho}m\,\cancel{\cos(\varphi)}+\varepsilon_0 c_0\varepsilon_{\mathrm{co}}\frac{A_H}{\varrho}\cancel{\cos(m\varphi)}\left[m-\frac{J_{m+1}(U)}{J_m(U)}U\right]\right)\\
H_\varphi^{\mathrm{cl}} &\overset{!}{=} K_{\mathrm{cl}}\left(\frac{n_{\mathrm{eff}}}{\varrho}m\,\cancel{\cos(\varphi)}+\varepsilon_0 c_0\varepsilon_{\mathrm{cl}}\frac{A_H}{\varrho}\cancel{\cos(m\varphi)}\left[m-\frac{K_{m+1}(W)}{K_m(W)}W\right]\right).
\end{aligned} \tag{3.13}$$

Again, solving for A_H yields

$$A_H = \frac{(K_{\mathrm{cl}}-K_{\mathrm{co}})n_{\mathrm{eff}}m}{\varepsilon_0 c_0\left[m(\varepsilon_{\mathrm{co}}-\varepsilon_{\mathrm{cl}})+K_{\mathrm{cl}}\varepsilon_{\mathrm{cl}}\dfrac{K_{m+1}(W)}{K_m(W)}W-K_{\mathrm{co}}\varepsilon_{\mathrm{co}}\dfrac{J_{m+1}(U)}{J_m(U)}U\right]}. \tag{3.14}$$

3.3 Dispersion relation

Determine the dispersion relation of the optical fiber by setting $A_E = A_H$.

Solution:

We first start by rewriting K_{co} and K_{cl} in terms of U and W

$$\begin{aligned}
U^2 &= \varrho^2(k_0^2\varepsilon_{\mathrm{co}}-\beta^2) \quad\Rightarrow\quad K_{\mathrm{co}} = \frac{k_0\varrho^2\mathrm{i}}{U^2}\\
W^2 &= \varrho^2(k_0^2\varepsilon_{\mathrm{cl}}-\beta^2) \quad\Rightarrow\quad K_{\mathrm{cl}} = -\frac{k_0\varrho^2\mathrm{i}}{W^2}.
\end{aligned} \tag{3.15}$$

Then we equate (3.12) and (3.14)

$$
\begin{aligned}
(K_{\text{cl}} - K_{\text{co}})^2 (n_{\text{eff}} m)^2 &= \left[K_{\text{cl}} \frac{K_{m+1}(W)}{K_m(W)} W - K_{\text{co}} \frac{J_{m+1}(U)}{J_m(U)} U \right] \\
&\quad \left[m(\varepsilon_{\text{co}} - \varepsilon_{\text{cl}}) + K_{\text{cl}} \varepsilon_{\text{cl}} \frac{K_{m+1}(W)}{K_m(W)} W - K_{\text{co}} \varepsilon_{\text{co}} \frac{J_{m+1}(U)}{J_m(U)} U \right] \\
\left(\frac{1}{U^2} + \frac{1}{W^2} \right)^2 (n_{\text{eff}} m)^2 &= \left[-\frac{K_{m+1}(W)}{W K_m(W)} - \frac{J_{m+1}(U)}{U J_m(U)} \right] \\
&\quad \left[m(\varepsilon_{\text{co}} - \varepsilon_{\text{cl}}) - \varepsilon_{\text{cl}} \frac{K_{m+1}(W)}{W K_m(W)} - \varepsilon_{\text{co}} \frac{J_{m+1}(U)}{U J_m(U)} \right] \\
\left(\frac{V}{UW} \right)^4 \left(\frac{n_{\text{eff}} m}{n_{\text{co}}} \right)^2 &= \left[\frac{K_{m+1}(W)}{W K_m(W)} + \frac{J_{m+1}(U)}{U J_m(U)} \right] \\
&\quad \left[m(\varepsilon_{\text{cl}} - \varepsilon_{\text{co}}) - \left(\frac{n_{\text{cl}}}{n_{\text{co}}} \right)^2 \frac{K_{m+1}(W)}{W K_m(W)} + \frac{J_{m+1}(U)}{U J_m(U)} \right]. \qquad (3.16)
\end{aligned}
$$

4 Weakly guiding fibers

4.1 Cut-off condition

Show that in weakly guidance approximation, the cut-off condition for the first higher-order mode is given by $0 = J_0(V)$.

Solution:

We can use the dispersion equation of weakly guided fiber modes

$$\frac{UJ_{l-1}(U)}{J_l(U)} = -\frac{WK_{l-1}(W)}{K_l(W)}, \tag{4.1}$$

where we consider the first order solution ($l = 1$). At the cut-off the modes get completely localized. This happens when $W^2 \propto \beta^2 - k_0^2 n_{\mathrm{cl}}^2 = 0$ and thus $U = V$. Then we find

$$\frac{UJ_0(U)}{J_1(U)} = 0 \quad \Rightarrow \quad J_0(V) = 0 \quad \text{for} \quad V \neq 0. \tag{4.2}$$

4.2 Cut-off numbers

Show that the cut-off numbers are given by $X_{lm} = (l + 2m)\frac{\pi}{2}$*, using the approximation*

$$J_l(X) = \sqrt{\frac{2}{\pi X}} \cos\left(X - \left(l + \frac{1}{2}\right)\frac{\pi}{2}\right), \tag{4.3}$$

which holds for large values of X *(multi-mode fiber).*

Solution:

From the first task the cut-off condition is given by $J_l(X) = 0$. Thus we set (4.3) to zero and find

$$\begin{aligned} 0 &= \cos\left(X - \left(l + \frac{1}{2}\right)\frac{\pi}{2}\right) \\ \Rightarrow m\pi + \frac{\pi}{2} &= X - \left(l + \frac{1}{2}\right)\frac{\pi}{2} \\ \Rightarrow X &= \frac{\pi}{2}\left(2m + l + \frac{3}{2}\right). \end{aligned} \tag{4.4}$$

For large values of X the term $3/2$ can be neglected compared to the other terms. This leads to

$$X = \frac{\pi}{2}(2m + l). \tag{4.5}$$

4.3 Effective index

Show that the propagation constant is analytic, using the definition of U and the assumption that $U = X_{lm}$ (vertical lines in dispersion function diagram).

Plot n_{eff} first as a function of m for $l = 0 (1 \leq m \leq 5)$ and second as function of l for $m = 1 (0 \leq l \leq 4)$ using $n_1 = 1.5, \varrho = 50\,\mu\text{m}, \lambda = 1\,\mu\text{m}$. What can you say about the phase velocity?

Solution:

We use the definition of U and try to solve for β

$$\frac{\pi}{2}(2m+l) = \varrho\sqrt{k_0^2 n_{\text{co}}^2 - \beta^2}$$

$$\beta^2 = k_0^2 n_{\text{co}}^2 - \frac{\pi^2}{4\varrho^2}(2m+l)^2. \tag{4.6}$$

Now using $\beta = n_{\text{eff}} k_0$ we can conclude

$$n_{\text{eff}} = \sqrt{n_{\text{co}}^2 - \left(\frac{\pi}{2\varrho k_0}\right)^2 (2m+l)^2}. \tag{4.7}$$

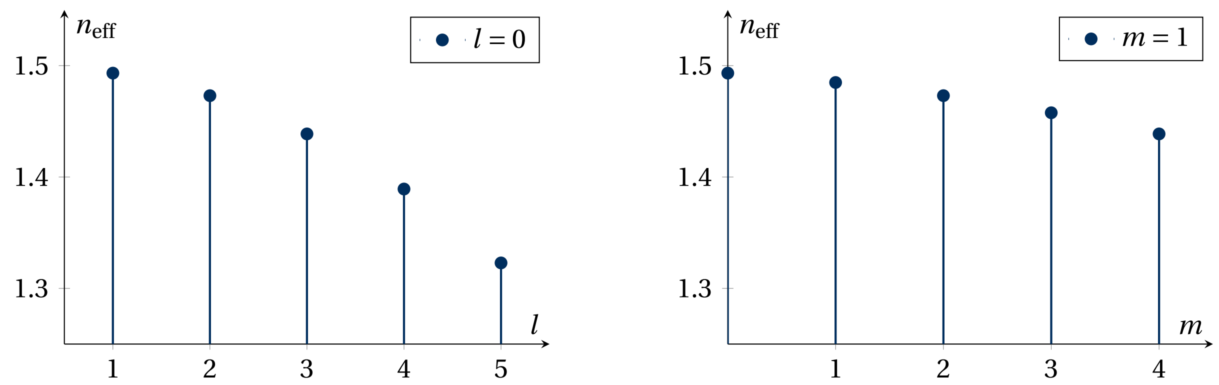

Fig. 6: Effective index of the weakly guided fiber for constant $l = 0$ (left) as a function of m and for constant $m = 1$ as a function of l.

By looking at figure 6 we conclude that the effective index decreases for higher order modes. Since the phase velocity $v_p = c/n_{\text{eff}}$ is inversly proportional to the effective index, it will increase for higher modes.

4.4 Group velocity

Derive an analytic expression for the group velocity, assuming that n_1 is wavelength independent.

Solution:

The group velocity is given as $v_g = \mathrm{d}\omega/\mathrm{d}\beta$. We start with (4.6) and take the total derivative

$$\begin{aligned} \beta^2 &= \frac{n_{\mathrm{co}}^2}{c^2}\omega^2 - \frac{\pi^2}{4\varrho^2}(2m+l)^2 \\ 2\beta\,\mathrm{d}\beta &= \frac{n_{\mathrm{co}}^2}{c^2}2\omega\,\mathrm{d}\omega \\ \Rightarrow v_g = \frac{\mathrm{d}\omega}{\mathrm{d}\beta} &= \frac{c}{n_{\mathrm{co}}^2}\frac{\beta}{k_0} = \frac{c}{n_{\mathrm{co}}^2}\sqrt{n_{\mathrm{co}}^2 - \left(\frac{\pi}{2\varrho k_0}\right)^2(2m+l)^2}. \end{aligned} \tag{4.8}$$

Since the group velocity is proportional to the effective index, it will decrease for higher order modes.

5 Pulse propagation

5.1 Pulse envelope without dispersion

Derive the pulse envelope for the case of vanishing group velocity dispersion (GVD) in the situation of a Gaussian pulse at input given by

$$F(t') = \mathrm{e}^{-\left(\frac{t}{\tau_p}\right)^2} \mathrm{e}^{-\mathrm{i}\omega_0 t}. \tag{5.1}$$

Solution:

We start with the Fourier analysis as done in the lecture

$$F(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{F}(z,\omega) \mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}\omega\,, \tag{5.2}$$

where $\tilde{F}(z,\omega)$ is the Fourier component of $F(z,t)$ at the frequency ω. In the frequency domain we know the spatial evolution of $\tilde{F}(z,\omega)$ namely

$$\tilde{F}(z,\omega) = \tilde{F}(0,\omega)\mathrm{e}^{\mathrm{i}\beta(\omega)z} \quad \text{with} \quad \tilde{F}(0,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(0,t)\mathrm{e}^{\mathrm{i}\omega t} \,\mathrm{d}t\,. \tag{5.3}$$

So then by substituting (5.3) into (5.2) we find

$$F(z,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(0,\omega)\mathrm{e}^{\mathrm{i}(\beta(\omega)z - \mathrm{i}\omega t)} \,\mathrm{d}\omega\,. \tag{5.4}$$

We start the calculations by first computing the Fourier transform of the input pulse using the Gaussian integral given as

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} \mathrm{e}^{\frac{b^2}{4a}}. \tag{5.5}$$

Then we find by using $\beta(\omega) = \beta_0 + \beta'(\omega - \omega_0)$

$$\tilde{F}(0,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\left(\frac{t}{\tau_p}\right)^2} \mathrm{e}^{\mathrm{i}(\omega-\omega_0)t} \,\mathrm{d}t = \tau_p \sqrt{\pi} \mathrm{e}^{-\frac{\tau_p^2}{4}(\omega-\omega_0)^2},$$

$$\Rightarrow F(z,t) = \frac{\tau_p\sqrt{\pi}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega\, \mathrm{e}^{-\frac{\tau_p^2}{4}(\omega-\omega_0)^2} \mathrm{e}^{\mathrm{i}[(\beta_0+\beta'(\omega-\omega_0))z-\omega t]}, \tag{5.6}$$

Now we perform a variable substitution $\bar{\omega} = \omega - \omega_0$ and find

$$F(z,t) = \frac{\tau_p\sqrt{\pi}}{2\pi} \mathrm{e}^{\beta_0 z - \mathrm{i}\omega_0 t} \int_{-\infty}^{\infty} \mathrm{d}\bar{\omega}\, \mathrm{e}^{-\frac{\tau_p^2}{4}\bar{\omega}^2} \mathrm{e}^{\mathrm{i}[\beta' z - t]\bar{\omega}} \tag{5.7}$$

$$= \exp\left(-\frac{(\beta' z - t)^2}{\tau_p^2}\right) \mathrm{e}^{\beta_0 z - \mathrm{i}\omega_0 t}. \tag{5.8}$$

5.2 Pulse envelope with dispersion

Derive the same envelope but now assume GVD ≠ 0.

Solution:

Now with group delay dispersion $\beta_2 \neq 0$ we write $\beta(\omega) = \beta_0 + \beta'(\omega - \omega_0) + \frac{1}{2}\beta''(\omega - \omega_0)^2$ and equation (5.7) modifies to

$$F(z,t) = \frac{\tau_p\sqrt{\pi}}{2\pi} \mathrm{e}^{\mathrm{i}(\beta_0 z - \omega_0 t)} \int_{-\infty}^{\infty} \mathrm{d}\bar{\omega}\, \mathrm{e}^{-\frac{\tau_p^2}{4}\bar{\omega}^2} \mathrm{e}^{\mathrm{i}\left[\frac{1}{2}\beta'' z\bar{\omega}^2 + (\beta' z - t)\bar{\omega}\right]}, \tag{5.9}$$

This is formally solved by setting $a = \frac{\tau_p^2}{4} - \frac{\mathrm{i}}{2}\beta'' z$ and $b = \mathrm{i}(\beta' z - t)$ in equation (5.5)

$$F(z,t) = \frac{\tau_p}{2} \frac{1}{\sqrt{\frac{\tau_p^2}{4} - \frac{\mathrm{i}}{2}\beta'' z}} \exp\left(-\frac{(\beta' z - t)^2}{\tau_p^2 + 2\mathrm{i}\beta'' z}\right) \mathrm{e}^{\mathrm{i}(\beta_0 z - \omega_0 t)} \tag{5.10}$$

5.3 Pulse width

Derive the position dependend pulse width and show that by using a pulse width level of $B = \mathrm{e}^{-1/4}$, the following equation results:

$$\tau(z) = \tau_p \sqrt{1 + \left(\frac{2z\beta_2}{\tau_p^2}\right)^2}. \tag{5.11}$$

Solution:

In order to find the pulse width we need to split the exponential in equation (5.10) into amplitude and phase

$$F(z,t) = \underbrace{\frac{1}{\sqrt{1 - \mathrm{i}\frac{2\beta_2 z}{\tau_p^2}}}}_{\text{amplitude change}} \underbrace{\exp\left(-\frac{(t - z\beta_1)^2\tau_p^2}{4z^2\beta_2^2 + \tau_p^4}\right)}_{\text{width change}} \underbrace{\exp\left(\mathrm{i}\frac{2z\beta_2(t - z\beta_1)}{4z^2\beta_2^2 + \tau_p^4}\right)}_{\text{local phase influence}} \underbrace{\mathrm{e}^{\mathrm{i}(\beta(\omega_0)z - \omega_0 t)}}_{\text{carrier phase}}. \tag{5.12}$$

We only need to consider the change of width described by the real part of the exponential term. For a pulse width level $B = \mathrm{e}^{-1/4}$ we demand

$$\frac{\tau_p^2}{4z^2\beta_2^2 + \tau_p^4} \stackrel{!}{=} \frac{1}{\tau(z)^2}. \tag{5.13}$$

Now we simply solve for $\tau(z)$

$$\tau(z) = \sqrt{\frac{4z^2\beta_2^2 + \tau_p^4}{\tau_p^2}} = \tau_p \sqrt{1 + \left(\frac{2z\beta_2}{\tau_p^2}\right)^2}. \tag{5.14}$$

5.4 Cross over position for different initial pulse lengths

Determine the cross-over position at which a pulse of initial width τ_p^a has a larger temporal pulse width than a pulse with ($\tau_p^b < \tau_p^a$). Assume the same GVD.

Solution:

Using (5.14) we can equate the two equations for different initial pulse lengths and solve for z

$$\begin{aligned}
\tau_p^a \sqrt{1 + \left(\frac{2z\beta_2}{\tau_p^a}\right)^2} &= \tau_p^b \sqrt{1 + \left(\frac{2z\beta_2}{\tau_p^b}\right)^2} \\
\Rightarrow (\tau_p^a)^2 - (\tau_p^b)^2 &= (2z\beta_2)^2 \left(\frac{1}{(\tau_p^b)^2} - \frac{1}{(\tau_p^a)^2}\right) \\
\Rightarrow z^2 &= \frac{1}{4\beta_2^2} \frac{(\tau_p^a)^2 - (\tau_p^b)^2}{\frac{1}{(\tau_p^b)^2} - \frac{1}{(\tau_p^a)^2}} \\
\Rightarrow z &= \frac{\tau_p^a \tau_p^b}{2\beta_2}. \qquad (5.15)
\end{aligned}$$