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Plasma Physics

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LaTeX-Satz und Design von Martin Beyer

Contents

1 Introduction

Books on plasma physics:

1. Francis F. Chen, “Introduction to Plasma Physics and Controlled Fusion, Volume 1: Plasma Physics”, Plenum Press (2016)
2. F. A. Bittencourt, “Fundamentals of Plasma Physics”, Springer (2004)
3. U. Schumacher, “Fusionsforschung”, Wissenschaftliche Buchgesellschaft (1993)

Many contents in this scripts and nearly all figures were taken from F. A. Bittencourt, “Fundamentals of Plasma Physics”.

1.1 Why study plasma physics?

The cheapest answer to the question state in the title of this section is that more than 99,99 % of all (visible) matter in the universe is in the *plasma state*.

But we should be more specific and try to define what a *plasma* actually is. Mostly a plasma is a very diluted, ionized gas, which consists of positive ions and electrons. However, we should be more specific but we will come to that later. The reason for that is, that even air contains ionized particles but is not considered to be a plasma.

Plasmas mainly occur in nature in astrophysical environments:

- Stellar interior (inside the sun)
- Stellar atmospheres
- Gaseous nebulas
- Interstellar hydrogen plasma
- Solar wind
- Van-Allen radiation belts
- Aurora borealis
- Lightning bolts

However, plasmas can also be generated on earth via technology:

- Gas inside a fluorescent light tube
- Nuclear fusion reactor
- Laser generated plasma

Luckily, we live in that $\ll 1\,\%$ of matter in the universe, in which plasmas do not occur naturally. The reason is, that the formation of complex molecules (that we are made of) is not possible in plasmas, since high energetic particles would collide and destroy them immediately.

1.2 The Saha equation

If we want to quantify the relative amount of ionized particles in a gas, we can use the SAHA equation, which we will simply state here:

$$\frac{n_i}{n_n} \approx 2.4 \cdot 10^{21} \frac{(T[\mathrm{K}])^{3/2}}{n_i[\frac{1}{\mathrm{m}^3}]} \exp\left(-\frac{U_i}{k_B T}\right), \tag{1.1}$$

where n_i and n_n are the number densities of ionized and neutral atoms. T denotes the gas temperature and U_i the ionization energy of the gas.

With the SAHA equation we can explain all relevant cases of plasma occurrences.

a) First we consider air at room temperature $T = 300\,\mathrm{K} \triangleq 25\,\mathrm{meV}$ and the ionization energy of nitrogen at $U_i \approx 14{,}5\,\mathrm{eV}$. With a number densities of gas molecules of about $n_n = 3 \cdot 10^{25}\,\frac{1}{\mathrm{m}^3}$ we find that the relative amount of ionized particles is

$$\frac{n_i}{n_i + n_n} \approx \frac{n_i}{n_n} \approx 10^{-122}. \tag{1.2}$$

b) The number of ionized atoms remains low until we reach a regime of $U_i \sim k_B T$, then n_i / n_n rises quickly. Then the gas is transformed into the plasma state mainly through collisional ionization. Eventually $n_i \gg n_n$, the plasma gets fully ionized. That is the reason why plasmas occur within stars ($T \geq 10^6 \ldots 10^7\,\mathrm{K}$) but not on earth.

c) Ionized atoms can recombine with electrons, but the recombination rate depends on $n_i \approx n_e$. This means that for low densities recombination is very unlikely and the degree of ionization remains high. Thus, the interstellar medium remains ionized/plasma even at very low temperatures.

1.3 Preliminary definition of a plasma

> A plasma needs to contain a large number of free electrons and ionized atoms or molecules. Nevertheless, a plasma can contain neutral particles and is macroscopically neutral. It also exhibits collective behaviour due to long-range COULOMB forces.

Often we observe other interactions involving neutral particles, which are only short-range forces (e. g. VAN-DER-WAALS forces in the gas, induced or permanent dipoles). These interactions only occur with direct neighbours (neutral-neutral oder neutral-charged). In contrast the interaction between two ore more charged particles has a much longer range of action. Thus, one charged particle interacts with a much larger number of particles. This leads to collective behaviour.

A more precise definition of a plasma is given in the following chapter via four criteria, if a certain medium containing charged particles acts as a plasma or not.

1.4 Plasma criteria

First we consider the effect of DEBYE shielding which is the distance over which influences of the electric field of a single particle is *felt* by other particles inside a plasma. It is given by:

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{n_e e^2}}, \tag{1.3}$$

where T is the electron temperature and n_e the electron number density. This equation describes the shielding of static electric fields on the order of λ_D via the plasma by rearranging itself. Thus, electric fields/potentials can occur in a plasma only over distances of λ_D.

The shielding effect also occurs at plasma boundaries leading to the formation of plasma sheaths around the boundary surface $d \approx \lambda_D$.

We can use the DEBYE length for the *first criterion* for the plasma definition:

> *The characteristic dimension L of the system must be larger than λ_D, otherwise there is not enough space for sufficient collective shielding*
>
> $$L \gg \lambda_D. \tag{1.4}$$

For the second criterion we define the *Debye-sphere* as the sphere with radius λ_D. The collective plasma behaviour between particles will only take place inside the DEBYE-sphere. The *second criterion* now states

> *The number of electrons within the* DEBYE*-sphere, $n_i \approx n_e$ is very large*
>
> $$N_D = \frac{4\pi}{3}\lambda_D^3 n_e \approx \boxed{n_e \lambda_D^3 \gg 1}. \tag{1.5}$$

We can also express this condition that the average distance between electrons $n_e^{-1/3}$ must be much lower than the DEBYE-length. Hereby we define the *plasma parameter* g as

$$g = \frac{1}{n_e \lambda_D^3} \quad \Rightarrow \quad \text{Plasma approximation:} \quad g \ll 1. \tag{1.6}$$

The *third criterion* states

> *The plasma is macroscopically neutral*
>
> $$\int_{V \gg \lambda_D^3} \left(n_e - \sum_i Z_i n_i \right) \mathrm{d}V \to 0, \tag{1.7}$$
>
> *where i describes all ion species in the plasma.*

The first three criteria of a plasma were based on a length scale of the plasma. Now we want to introduce a characteristic time/frequency scale called the *plasma frequency*. The instantaneous disturbance of the plasma from the equilibrium condition leads to local charge separations and space charge fields. They are oriented in such a way, that they pull back the electrons to their initial positions. This leads to an *overshoot* of the electrons and creates an oscillation around the equilibrium position with a natural frequency

$$\omega_p = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}} \quad \text{plasma frequency} \quad . \tag{1.8}$$

In the temporal average we will still preserve macroscopic charge neutrality. However, on time scales

$$T_p = \frac{2\pi}{\omega_p}, \tag{1.9}$$

dissipation effects need to be taken into account. For example collisions between electrons and neutral particles lead to damping. If they occur with a characteristic frequency $\nu_{en} = \frac{1}{\tau_{en}}$ the interaction with neutral particles dominates, if $\nu_{en} > \omega_p$. Then the plasma behaves like a neutral gas. This leads us to the *fourth criterion*

> *The average time τ between $e-n$ collisions has to be large compared to the characteristic time over which the plasma parameters are changing*
>
> $$\boxed{\omega\tau_{en} > 1}, \tag{1.10}$$
>
> *where ω is the angular frequency of typical plasma oscillations (e. g. the plasma frequency).*

This means not only the electron density n_e and temperature T_e, but also the density of neutral particles n_n is important for the plasma definition.

1.5 Application of plasma physics to thermonuclear fusion

First we want to note, that the binding energy per nucleon E_{bind} depends on the atomic mass number. The most stable nucleus is ^{56}Fe. If we use nuclear reactions like fusion we can gain energy by e. g. burning ^{1}H to ^{4}He in the sun. The chemical reaction looks like

$$\begin{aligned} &4\,^1\text{H} \rightarrow \ldots \rightarrow \,^4\text{He} + \Delta E \\ &\Delta E = \Delta m c^2 = (4 \cdot 1.008145\text{u} - 4.00387\text{u})c^2 = 26{,}72\,\text{MeV}. \end{aligned} \tag{1.11}$$

This process happens in the sun at a rate of

$$\approx 600 \cdot 10^9\,\text{kg}\,^1\text{H} \rightarrow 596 \cdot 10^9\,\text{kgHe} + \frac{\Delta E}{c^2} \tag{1.12}$$

every single second. This energy difference corresponds to an intensity of $1{,}37\,\frac{\text{kW}}{\text{m}^2}$ on the surface of the earth. The proton-proton reaction in the sun occurs in different steps namely

1. $^1H + ^1H \longrightarrow ^2He \longrightarrow ^2H + e^+ + \nu_e + 0{,}42\,\text{MeV}$
2. $e^- + e^+ \longrightarrow 2\gamma\,(1{,}022\,\text{MeV})$
3. $^2H + ^1H \longrightarrow ^3He + \gamma\,(5{,}49\,\text{MeV})$
4. $^3He + ^3He \longrightarrow ^4He + 2\,^1H\,(2{,}86\,\text{MeV})$

In total the p-p reaction looks like

$$6\,^{H} \longrightarrow {}^4He + 2\,^1H + 2\,e^+ + 2\,\nu_e + 2\,\gamma + (26{,}724\,\text{MeV}). \tag{1.13}$$

This is the dominant reaction for the temperatures in the sun at $T = 10\ldots10^6\,\text{K}$. We may ask, why we cannot utilize this reaction on earth. The problem lies within the first step where two Hydrogen atoms form the Deuterium atom. Here the cross section (due to weak interaction) is very low, thus this process is not feasible on earth.

Other possible fusion reactions may be:

- $^2H + ^2H \longrightarrow ^3He + n + 3{,}27\,\text{MeV}$
- $^2H + ^2H \longrightarrow ^3H + ^1H + 4{,}03\,\text{MeV}$
- $^2H + ^3H \longrightarrow ^4He + n + 17{,}58\,\text{MeV}$
- $^2H + ^3He \longrightarrow ^4He + ^1H + 18{,}34\,\text{MeV}$

The third fusion process is to be considered the best reaction to realize nuclear fusion on the earth.

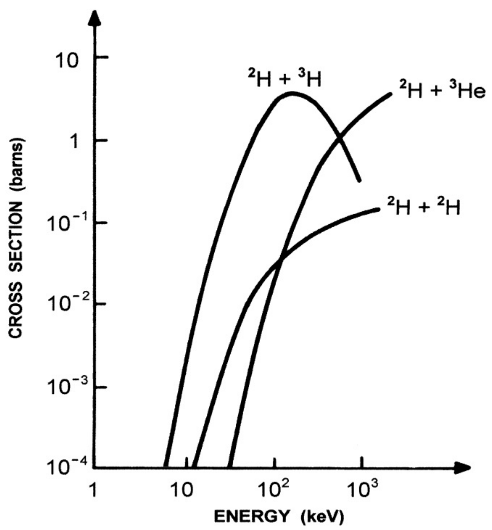


Fig. 1: Fusion cross sections, in barns (1 barn = $10^{-28}\,\text{m}^2$), as a function of energy, in keV, for the hydrogen reactions.

We also need to consider the cross sections of the fusion reactions shown in figure 1. The challenge is that we need to generate plasmas at high temperatures (~ 10 keV) to surpass the COULOMB barrier of the positively charged nuclei. We also need to have a high density in order to keep the fusion reaction at high temperatures. Furthermore the plasma needs to be confined for a sufficiently long time τ. Then we have a substantial number of fusion reactions which leads to ignition. For this to work the plasma confinement is essential.

One criterion for ignition of nuclear fusion is called the *Lawson criterion*

$$n_e \tau \cdot k_B T \approx 3{,}3 \cdot 10^{15} \, \frac{\mathrm{keV\,s}}{\mathrm{cm}^3}, \tag{1.14}$$

which applies to all kinds of plasma confinement. We also want to mention different confinement types:

1. open systems (magnetic mirrors/bottles)
2. closed systems (toruses)
3. laser-pellet fusion, Inertial Confinement Fusion ICF

The first two confinement types are referred as Magnetic Confinement Fusion MCF.

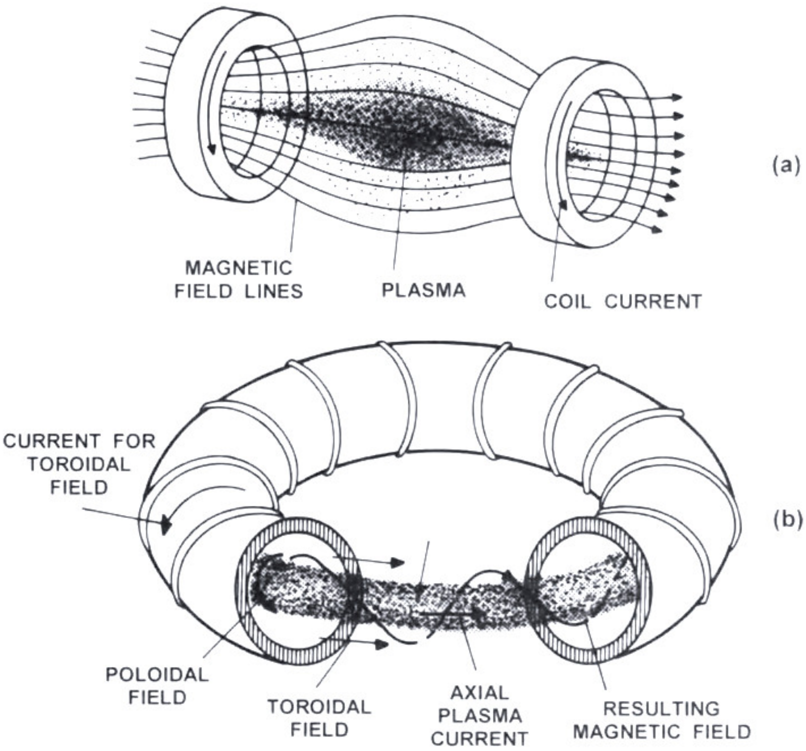


Fig. 2: Schematic illustration showing the magnetic field configurations of some basic schemes for plasma confinement. (a) magnetic mirror system. (b) Tokamak.

2 Motion of charged particles in constant and uniform fields

2.1 Preconditions

We assume that $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ as a function of space $\boldsymbol{r}$ and time t, not affected by particles. In this chapter we assume, that the fields are temporally constant and spatially uniform. The motion of charged particles in the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ is governed by the LORENTZ equation

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}. \tag{2.1}$$

In this lecture course we will neglect any relativistic effects concerning the change of the mass of the charged particles, thereby $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$. We will also neglect radiation effects caused by the acceleration of the charged particles.

2.2 Energy conservation

We want to discuss energy conservation for two different cases. We start with the situation of $\boldsymbol{E} = 0$. Then

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} &= q\boldsymbol{v} \times \boldsymbol{B} \\ m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v} &= q(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} = 0 \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\boldsymbol{v}^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} = 0. \end{aligned} \tag{2.2}$$

The kinetic energy of the charged particle is constant. The $\boldsymbol{B}$-field alone does not transfer energy to the particle. This relation is not valid for $\frac{\partial \boldsymbol{B}}{\partial t} \neq 0$ because then, due to FARADAY's law, an energy transfer is possible.

For a nonzero electric field $\boldsymbol{E} \neq 0$ we find

$$\begin{aligned} m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{v} &= q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\boldsymbol{v}^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} = q\boldsymbol{E} \cdot \boldsymbol{v}. \end{aligned} \tag{2.3}$$

Since $\vec{\nabla} \times \boldsymbol{E} = 0$ the electric field can be written as the gradient of the electrostatic potential

$$\boldsymbol{E} = -\vec{\nabla}\Phi_{\mathrm{el}} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{kin}} = -q\vec{\nabla}\Phi_{\mathrm{el}} \cdot \boldsymbol{v} = -q\frac{\mathrm{d}\Phi_{\mathrm{el}}}{\mathrm{d}t}. \tag{2.4}$$

For a static potential Φ_{el} that is not explicitly dependent on time we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}mv^2 + q\Phi_{\mathrm{el}}\right] = 0. \tag{2.5}$$

Thus the sum of kinetic and electric potential energy is constant for static electric and magnetic fields.

2.3 Uniform electric and magnetic field

Uniform E-field

We start with the situation of vanishing $\boldsymbol{B}$-field and $\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0$. Then the equation of motion is simplified to

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q\boldsymbol{E}_0 \quad \Rightarrow \quad \boldsymbol{r}(t) = \frac{1}{2}\frac{q\boldsymbol{E}_0}{m}t^2 + \boldsymbol{v}_0 t + \boldsymbol{r}_0, \tag{2.6}$$

where $\boldsymbol{r}_0$ and $\boldsymbol{v}_0$ are the initial position and velocity at $t = 0$. We observe an acceleration $\frac{q\boldsymbol{E}_0}{m}$ only in the direction of the $\boldsymbol{E}$-field.

Uniform B-field

Now the situation is reversed with vanishing $\boldsymbol{E}$-field and $\boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}_0$. We split the particles velocity $\boldsymbol{v}$ into components parallel $\boldsymbol{v}_\parallel$ and perpendicular $\boldsymbol{v}_\perp$ with respect to $\boldsymbol{B}_0$. Then the equation of motion becomes

$$\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{v}_\perp}{\mathrm{d}t} = \frac{q}{m}(\boldsymbol{v}_\perp \times \boldsymbol{B}_0). \tag{2.7}$$

The force due to the perpendicular velocity component does not affect the parallel component since $\boldsymbol{v}_\perp \times \boldsymbol{B}_0$ is perpendicular to $\boldsymbol{B}_0$ and thus also perpendicular to $\boldsymbol{v}_\parallel$. This indicates that the parallel motion remains unchanged $\boldsymbol{v}_\parallel = \boldsymbol{v}_{\parallel,0}$.

However, for the perpendicular component we find

$$\frac{\mathrm{d}\boldsymbol{v}_\perp}{\mathrm{d}t} = \boldsymbol{\omega}_c \times \boldsymbol{v}_\perp \quad \text{with} \quad \boldsymbol{\omega}_c = -\frac{q\boldsymbol{B}_0}{m} = \frac{|q|B_0}{m}\boldsymbol{e}_\omega. \tag{2.8}$$

The vector $\boldsymbol{\omega}_c$ is a constant *axial* vector pointing in direction of $\boldsymbol{B}$ for $q < 0$ and in opposite direction of $\boldsymbol{B}_0$ for $q > 0$. Integrating equation (2.8) leads us to

$$\boldsymbol{v}_\perp = \boldsymbol{\omega}_c \times \boldsymbol{r}_c \tag{2.9}$$

with $\boldsymbol{r}_c$ being a vector pointing from the (momentary) centre of gyration $\boldsymbol{G}$ to the momentary position of the particle. We call $\boldsymbol{G}$ the guiding centre of the particles motion.

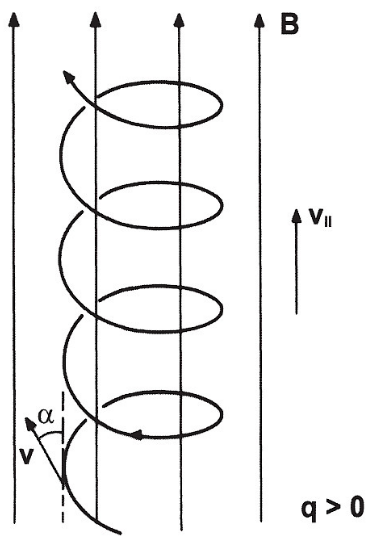
For $\boldsymbol{v}_\parallel \neq 0$ both trajectories form a helix shape with a pitch angle α

$$\alpha = \arctan\left(\frac{v_\perp}{v_\parallel}\right) = \arcsin\left(\frac{v_\perp}{v}\right). \tag{2.10}$$

The length ω_c of the axial gyration vector is called *cyclotron* or *Larmor* frequency and r_c

$$r_c = \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B_0} \tag{2.11}$$

is called the *Larmor* radius.



Uniform $\boldsymbol{E}$ and $\boldsymbol{B}$-field

If we have both uniform electric and magnetic fields we again separate $\boldsymbol{v}$ and $\boldsymbol{E}$ into components parallel and perpendicular to $\boldsymbol{B}_0$. Then we find for the equations of motion

$$m\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = q\boldsymbol{E}_\parallel \tag{2.12}$$

$$m\frac{\mathrm{d}\boldsymbol{v}_\perp}{\mathrm{d}t} = q(\boldsymbol{E}_\perp + \boldsymbol{v}_\perp \times \boldsymbol{B}). \tag{2.13}$$

Both equations are evolving completely independently. We can solve the first equation (2.12) immediately as

$$\boldsymbol{r}_\parallel = \frac{1}{2}\left(\frac{q\boldsymbol{E}_\parallel}{m}\right)t^2 + \boldsymbol{v}_{\parallel,0}\,t + \boldsymbol{r}_0. \tag{2.14}$$

The particle is accelerated *along* the magnetic field lines by the $\boldsymbol{E}_\parallel$-field. For the second equation (2.13) we make the ansatz

$$\boldsymbol{v}_\perp = \boldsymbol{v}'_\perp(t) + \boldsymbol{v}_E \tag{2.15}$$

where $\boldsymbol{v}_E$ is a constant velocity in the plane perpendicular to $\boldsymbol{B}$ which is still to be determined. However, if we transform into a frame of reference moving with $\boldsymbol{v}_E$, $\boldsymbol{v}'_\perp$ describes the particles velocity perpendicular to $\boldsymbol{B}$.

Now we will rewrite the perpendicular component $\boldsymbol{E}_\perp$ of the electric field as

$$\boldsymbol{E}_\perp = -\frac{\boldsymbol{E}_\perp \times \boldsymbol{B}}{B^2} \times \boldsymbol{B} \tag{2.16}$$

and using $\frac{\mathrm{d}\boldsymbol{v}_E}{\mathrm{d}t} = 0$ we can rewrite (2.13) as

$$\begin{aligned} m\frac{\mathrm{d}\boldsymbol{v}'_\perp}{\mathrm{d}t} &= q\boldsymbol{E}_\perp + (\boldsymbol{v}'_\perp + \boldsymbol{v}_E) \times \boldsymbol{B} \\ &= q\left(-\frac{\boldsymbol{E}_\perp \times \boldsymbol{B}}{B^2} + (\boldsymbol{v}'_\perp + \boldsymbol{v}_E)\right) \times \boldsymbol{B}. \end{aligned} \tag{2.17}$$

If we choose $\boldsymbol{v}_E = \frac{\boldsymbol{E}_\perp \times \boldsymbol{B}}{B^2}$ the equation simplifies to

$$m\frac{\mathrm{d}\boldsymbol{v}'_\perp}{\mathrm{d}t} = q(\boldsymbol{v}'_\perp \times \boldsymbol{B}). \tag{2.18}$$

In this frame of reference the electric field component is “transformed away” and $\boldsymbol{B}$ remains unchanged. Here the particle performs a circular motion with the LARMOR frequency $\boldsymbol{\omega}_c$ and radius $\boldsymbol{r}'_c$. The overall motion is now a superposition of a

1. circular motion (cyclotron motion) in the plane $\perp$ to $\boldsymbol{B}$
2. uniform motion with $\boldsymbol{v}_E \perp \boldsymbol{E}_\perp$ and $\boldsymbol{B}$
3. uniform acceleration with $\frac{q\boldsymbol{E}_\parallel}{m}$ along $\boldsymbol{B}$

Total velocity of charged particles in uniform fields

$$\boldsymbol{v}(t) = \boldsymbol{\omega}_c \times \boldsymbol{r}_c + \frac{\boldsymbol{E}_\perp \times \boldsymbol{B}}{B^2} + \frac{q\boldsymbol{E}_\parallel}{m} + \boldsymbol{v}_\parallel(0). \tag{2.19}$$

Since $\boldsymbol{E}_\parallel \times \boldsymbol{B} = 0$, the constant velocity $\boldsymbol{v}_E$ becomes

$$\boldsymbol{v}_E = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} \qquad \text{drift velocity.} \tag{2.20}$$

It is called the uniform $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity of the guiding centre of the gyration. It is independent of mass and sign of the charge of the particles, i. e. it is the same for all charged particles. This leads to a cycloid motion in the plane perpendicular to $\boldsymbol{B}$ in the lab frame.

Physical explanation of the drift velocity

We also want to give a physical explanation of the $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity as shown in figure 3. We note that $\boldsymbol{E}_\perp$ and $\boldsymbol{B}$ act simultaneously on the charged particles. $E_\perp$ accelerates the particles, if the $\boldsymbol{B}$-field induced cyclotron motion is parallel to the electric field $\boldsymbol{E}_\perp$ (for $q > 0$), whereas it decelerates them when the motion is antiparallel to $\boldsymbol{E}_\perp$. If we now look at the Larmor radius

$$r_c = \frac{m}{q}\frac{v_\perp}{B}, \tag{2.21}$$

we observe that it changes accordingly under the action of $\boldsymbol{E}_\perp$. So as the particle is accelerated, the radius of curvature increases and in the next half cycle it decreases again. This leads to a drift of the guiding centre perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$. We observe a different number of arcs for the (heavy) ions and (light) electrons. However, the size of the arcs is larger for the heavy ions. Thus the net drift velocity $\boldsymbol{v}_E$ is the same for all charged particle species in a plasma. For a collisionless plasma we will not observe a net current, since all particles move in the same direction with equal speed.

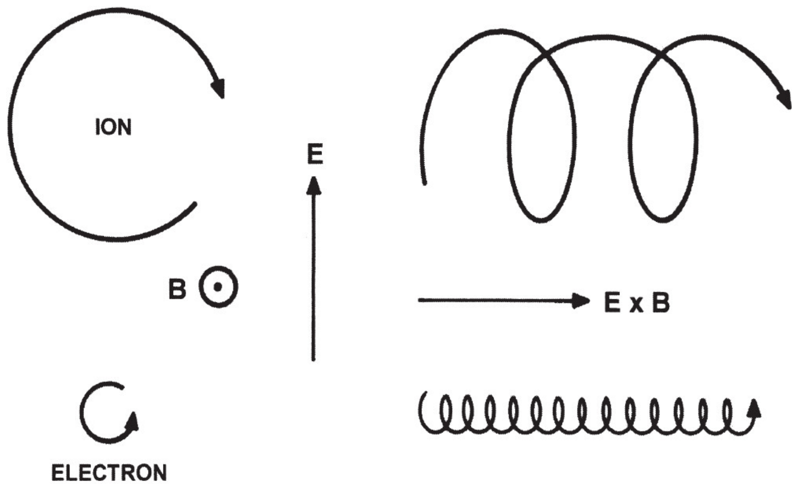


Fig. 3: Cycloidal trajectories by ions and electrons in crossed electric and magnetic fields. We observe a drift in the direction of $\boldsymbol{E} \times \boldsymbol{B}$.

However, if collisions e. g. with neutral particles are taken into account, the drift velocities for ions and electrons are different (ions experience more collisions). Then the ion current is smaller than the electron current leading to a net current perpendicular to $\boldsymbol{E}$ and $\boldsymbol{B}$. Since this current is mainly produced by the electrons, it is oriented antiparallel to the electron motion. This current is known as *Hall current*.

2.4 Drift due to external forces

We assume an additional, uniform and constant force $\boldsymbol{F}$ acting on the plasma, e. g. gravitation. Then the equations of motion are modified as

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + \boldsymbol{F}. \tag{2.22}$$

This leads to another drift velocity analogous to the effect of the electric field $\boldsymbol{E}$

$$\boldsymbol{v}_F = \frac{\boldsymbol{F} \times \boldsymbol{B}}{qB^2} = \frac{m}{q}\frac{\boldsymbol{g} \times \boldsymbol{B}}{B^2} \quad \text{for} \quad \boldsymbol{F} = m\boldsymbol{g}. \tag{2.23}$$

Note that $\boldsymbol{v}_F$ depends on $\frac{m}{e}$. For oppositely charged particles this leads to different drift directions. This leads to a non-vanishing current, even for collisionless plasma.

3 Charged particles in non-uniform magnetostatic fields

3.1 Preconditions

The solution of the equation of motion is often difficult for varying fields, sometimes analytical solutions do not exist and we need to implement numerical schemes. However, we can make approximations if

- the details of the motion are not of interest (only drift motion is of interest)
- B-field is *strong* and slowly varying in space
- E-field is weak (or vanishing).

Therefore we can assume that the fields are constant/uniform with respect to one gyration about $\boldsymbol{B}$. Here the particles move in static and slightly inhomogeneous $\boldsymbol{B}$-fields. Slightly inhomogeneous means that the variation δB of the magnitude of $\boldsymbol{B}$ over r_c is much smaller than the amplitude of the magnetic field

$$\delta B = r_c \vec{\nabla}|\boldsymbol{B}| << |\boldsymbol{B}|. \tag{3.1}$$

Then a solution in first-order approximation is sufficient. This is often referred as the ALFVÉN-approximation or *guiding-centre-approximation.*

For homogeneous magnetic fields $\boldsymbol{B}$ the motion is perfectly circular and the guiding centre moves with constant speed along $\boldsymbol{B}$. For slightly inhomogeneous magnetic fields the motion about $\boldsymbol{B}$ is nearly circular leading to a drift across $\boldsymbol{B}$ and a gradual change of the velocity component along the magnetic field lines.

In this description the gyration is not of interest, we only consider the guiding centre motion corresponding to an average over the rapid gyrations. Thus we are looking for transverse drift and the parallel acceleration of the guiding centre with respect to $\boldsymbol{B}$.

3.2 Spatial variation of the magnetic field

We start with a general magnetic field in cartesian coordinates

$$\boldsymbol{B}(\boldsymbol{r}) = B_x(\boldsymbol{r})\hat{\mathbf{e}}_x + B_y(\boldsymbol{r})\hat{\mathbf{e}}_y + B_z(\boldsymbol{r})\hat{\mathbf{e}}_z. \tag{3.2}$$

Here each component $B_i(\boldsymbol{r})$ may vary with respect to all three cartesian coordinates. This means that nine parameters are necessary for a complete description of the spatial variation of $\boldsymbol{B}$ at a certain point $\boldsymbol{r}$:

$$\vec{\nabla}\boldsymbol{B} = (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z) \begin{pmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial x} & \frac{\partial B_z}{\partial x} \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{e}}_x \\ \hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z \end{pmatrix} = (\vec{\nabla} B_x, \vec{\nabla} B_y, \vec{\nabla} B_z). \tag{3.3}$$

Here, the gradient is applied to the different vector components of the magnetic field. Strictly speaking, this quantity is a tensor. Since

$$\vec{\nabla} \cdot \boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{3.4}$$

only 8 terms (or two divergence terms) are independent. The number of independent terms is further restricted by AMPERÉS law

$$\vec{\nabla} \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{j} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \tag{3.5}$$

if the current density $\boldsymbol{j}$ is nonzero or we have temporal varying electric fields. For $\boldsymbol{B}(\boldsymbol{r} = 0) = \boldsymbol{B}_0 = B_0 \hat{\mathbf{e}}_z$ we can sort the different terms of the tensor in equation 3.3 into different categories:

- divergence terms: $\frac{\partial B_x}{\partial x}, \frac{\partial B_y}{\partial y}, \frac{\partial B_z}{\partial z}$
- gradient terms: $\frac{\partial B_z}{\partial x}, \frac{\partial B_z}{\partial y}$
- curvature terms: $\frac{\partial B_x}{\partial z}, \frac{\partial B_y}{\partial z}$
- shear terms: $\frac{\partial B_x}{\partial y}, \frac{\partial B_y}{\partial x}$

Now we may ask how the $\boldsymbol{B}$-field looks like due to influences of the different groups.

Divergence terms

Assuming that $\frac{\partial B_z}{\partial z} \neq 0$, at least one of the other divergence terms $\frac{\partial B_y}{\partial y}, \frac{\partial B_z}{\partial z}$ cannot vanish either due to $\vec{\nabla} \cdot \boldsymbol{B} = 0$. At any point in space the magnetic flux lines are parallel to $\boldsymbol{B}$ and their areal density is proportional to $|\boldsymbol{B}|$ at this point. Then $\boldsymbol{B}$ is also referred to as the magnetic flux density.

We can now define an element of arc along such a flux line $\mathrm{d}\boldsymbol{s} = \mathrm{d}x\,\hat{\mathbf{e}}_x + \mathrm{d}y\,\hat{\mathbf{e}}_y + \mathrm{d}z\,\hat{\mathbf{e}}_z$. Since $\mathrm{d}\boldsymbol{s} \parallel \boldsymbol{B}$

$$\mathrm{d}\boldsymbol{s} \times \boldsymbol{B} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}x}{B_x} = \frac{\mathrm{d}y}{B_y} = \frac{\mathrm{d}z}{B_z}. \tag{3.6}$$

Since around the origin $\boldsymbol{B}$ mainly varies in $\hat{\mathbf{e}}_z$ direction we can expand B_x and B_y into a TAYLOR series up to first order

$$\begin{aligned} B_x(x,0,0) &\approx B_x(0,0,0) + \frac{\partial B_x}{\partial x} x_1 = \frac{\partial B_x}{\partial x} x_1 \\ B_y(y,0,0) &\approx B_y(0,0,0) + \frac{\partial B_y}{\partial y} y_1 = \frac{\partial B_y}{\partial y} y_1. \end{aligned} \tag{3.7}$$

Now lets look at the projections of the magnetic field lines crossing the (x,y)-plane at $z=0$ onto the (x,z)-plane and (y,z)-plane shown in figure 4.

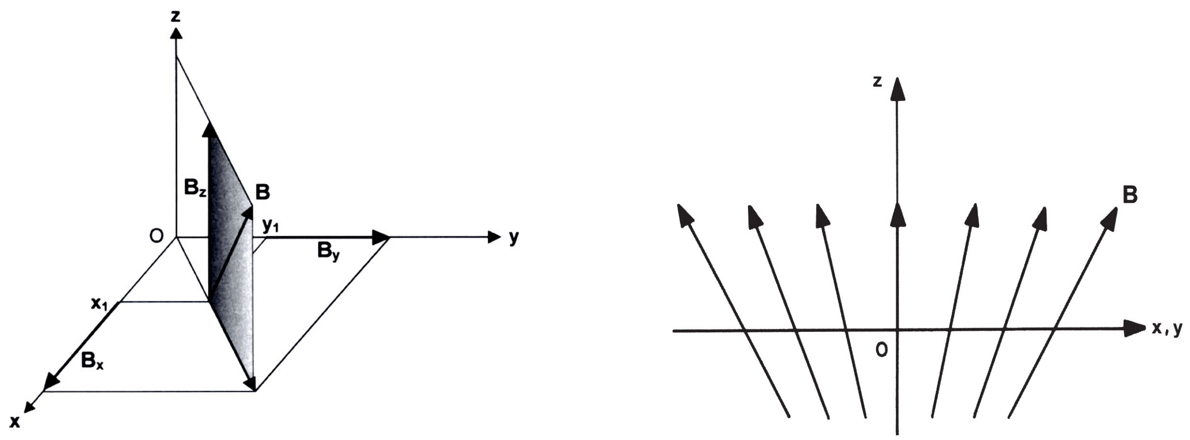


Fig. 4: Left: Projections of different components of $\boldsymbol{B}$-field onto the three axes of the cartesian coordinate system.
Right: Geometry of the magnetic field lines corresponding to the divergence terms, when they are positive.

Then we find the following relations

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{B_x}{B_z} = \frac{1}{B_z}\frac{\partial B_x}{\partial x} x_1 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{B_y}{B_z} = \frac{1}{B_z}\frac{\partial B_y}{\partial y} y_1. \tag{3.8}$$

This means that the field lines have to converge or diverge in (x,z)-plane or (y,z)-plane depending on the sign of the divergence terms of $\boldsymbol{B}$. This is one of the reasons, why the vector operation $\vec{\nabla}\cdot\boldsymbol{B}$ is called divergence.

Gradient and curvature terms

Here we assume that the $\boldsymbol{B}$-field magnitude increase in $\hat{\mathbf{e}}_x$ direction. This is schematically shown in figure 5.

We first may assume that the z-component of $\boldsymbol{B}$ may change linearly in x-direction like

$$\boldsymbol{B} = B_0(1 + a\,x)\hat{\mathbf{e}}_z. \tag{3.9}$$

However, for $\boldsymbol{j} = 0$ this equation cannot fulfil $\vec{\nabla}\times\boldsymbol{B} = 0$. Therefore we have to add a curvature term like

$$\boldsymbol{B} = B_0(a\,z\hat{\mathbf{e}}_x + (1 + a\,x)\hat{\mathbf{e}}_z). \tag{3.10}$$

Thus a gradient term is only possible in combination with the curvature of the magnetic field lines.

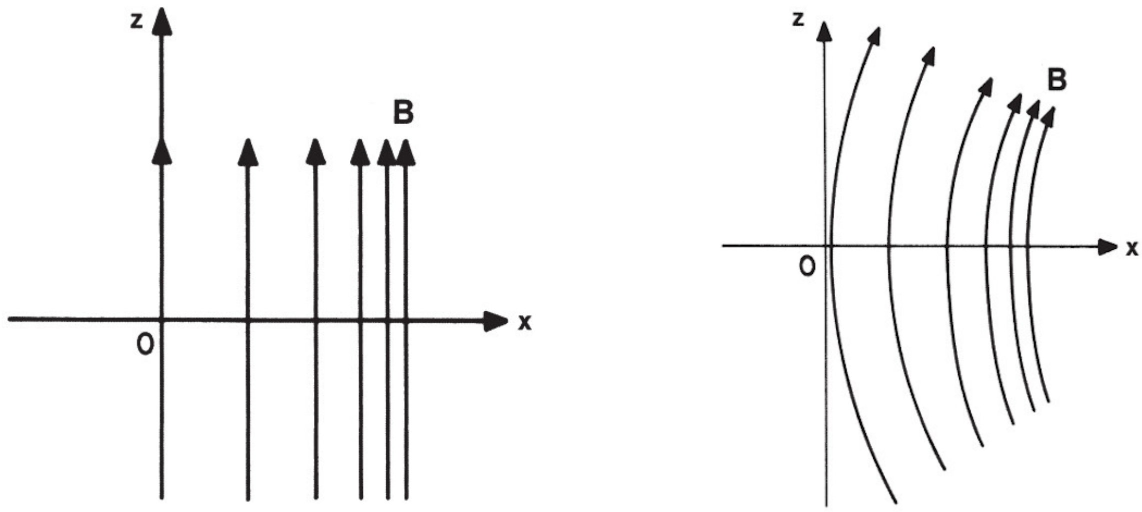


Fig. 5: Left: Geometry of the magnetic field lines corresponding to equation (3.9) when $\boldsymbol{B}$ has a gradient in the x direction. However, this field does not satisfy $\vec{\nabla} \times \boldsymbol{E} = 0$.
Right: Geometry of the magnetic field lines corresponding to (3.10), with gradient and curvature terms.

Shear terms

Shear terms lead to twisting of the magnetic field lines about each other. However, no first order drift velocity is caused by these terms. Therefore we will neglect them in our discussion.

3.3 Equation of motion in first order approximation

Again we assume now that the magnetic field in the origin has the form $\boldsymbol{B}(\boldsymbol{r}=0) = B_0\hat{\mathbf{e}}_z$. In the proximity of the origin we can expand the magnetic field into a first order TAYLOR expansion

$$\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{B}_0 + \boldsymbol{r}\cdot(\vec{\nabla}\boldsymbol{B}) + \dots, \tag{3.11}$$

where $\vec{\nabla}\boldsymbol{B}$ denotes the tensor given in (3.3). We may also rewrite this as

$$\boldsymbol{r}\cdot(\vec{\nabla}\boldsymbol{B}) = (\boldsymbol{r}\cdot\vec{\nabla})\boldsymbol{B} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\boldsymbol{B}, \tag{3.12}$$

where nine partial derivatives of $\boldsymbol{B}$ need to be calculated at the origin. Since we are assuming that the spatial variation of $\boldsymbol{B}$ in the order of the LARMOR radius is negligible, the condition

$$\delta B = |\boldsymbol{r}\cdot(\vec{\nabla}\boldsymbol{B})| \ll |\boldsymbol{B}_0| \tag{3.13}$$

is clearly met. We can now write down the equations of motion with $\boldsymbol{E} = 0$ using (3.11)

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{v}\times\boldsymbol{B}_0) + q\boldsymbol{v}\times[\boldsymbol{r}\cdot(\vec{\nabla}\boldsymbol{B})]. \tag{3.14}$$

The actual particle velocity can also be written as a superposition

$$\boldsymbol{v} = \boldsymbol{v}^{(0)} + \boldsymbol{v}^{(1)} = \frac{\mathrm{d}\boldsymbol{r}^{(0)}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{r}^{(1)}}{\mathrm{d}t}, \tag{3.15}$$

where $|\boldsymbol{v}^{(1)}| \ll |\boldsymbol{v}^{(0)}|$ is a first-order approximation. The solution of the zero-order equation

$$m\frac{\mathrm{d}\boldsymbol{v}^{(0)}}{\mathrm{d}t} = q(\boldsymbol{v}^{(0)} \times \boldsymbol{B}_0) \tag{3.16}$$

was already done in chapter 2. Neglecting all second order terms (any product of first order terms $\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{B}$) we find

$$\boldsymbol{v} \times [\boldsymbol{r} \cdot (\vec{\nabla}\boldsymbol{B})] = \boldsymbol{v}^{(0)} \times [\boldsymbol{r}^{(0)} \cdot (\vec{\nabla}\boldsymbol{B})]. \tag{3.17}$$

Including all these approximations, the equation of motion (3.14) becomes

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{v} \times \boldsymbol{B}_0) + q\boldsymbol{v}^{(0)} \times [\boldsymbol{r}^{(0)} \cdot (\vec{\nabla}\boldsymbol{B})]. \tag{3.18}$$

The second term on the right hand side refers to an additional force term which we already discussed in section 2.4. This force, however, is not constant since it depends on the instantaneous particle position $\boldsymbol{r}^{(0)}$. This leads to small oscillations during the gyration period. We can smooth out the oscillations by averaging this force term over one gyration period of the force term $q\boldsymbol{v}^{(0)} \times [\boldsymbol{r}^{(0)} \cdot (\vec{\nabla}\boldsymbol{B})]$. This allows us to determine the *parallel acceleration* of the guiding centre and its *transverse drift*.

3.4 Average force over one gyration period

Now we consider the case where the component of the particle's initial velocity along the $\boldsymbol{B}$-field is zero, so that the particle's path is nearly circular. In a uniform magnetic field this would be equivalent to observing the particle motion in a coordinate system moving with the guiding centre velocity $\boldsymbol{v}_\parallel \neq 0$. However, when the field lines are bent, a frame of reference sliding along $\boldsymbol{B}$ is not an inertial system anymore and the curvature of the field lines give rise to *inertial forces* and therefore a *curvature drift* of the particle.

For the moment we will assume that the field lines are *straight* and the frame of reference is moving with $\boldsymbol{v}_\parallel$ and we only observe transverse motion. The zero-order variables $\boldsymbol{r}^{(0)}$ and $\boldsymbol{v}^{(0)}$ are situated in the (x, y)-plane. The force term from last section

$$\boldsymbol{F} = q\boldsymbol{v}^{(0)} \times [\boldsymbol{r}^{(0)} \cdot (\vec{\nabla}\boldsymbol{B})] \tag{3.19}$$

can be separated into components parallel $\boldsymbol{F}_\parallel$ and perpendicular $\boldsymbol{F}_\perp$ with respect to $\boldsymbol{B} = B_0\hat{\mathbf{e}}_z$. Using a local cylindrical coordinate system (r, φ, z) we have

$$\boldsymbol{r}^{(0)} \cdot (\vec{\nabla}\boldsymbol{B}) = r^{(0)}\frac{\partial \boldsymbol{B}}{\partial r}. \tag{3.20}$$

Any component of $(\vec{\nabla}\boldsymbol{B})$ perpendicular to $\boldsymbol{r}^{(0)}$ cancels due to the dot-product. For $\boldsymbol{B} = B_r\hat{\mathbf{e}}_r + B_\varphi\hat{\mathbf{e}}_\varphi + B_z\hat{\mathbf{e}}_z$, the φ-component is parallel to $\boldsymbol{v}^{(0)}$ (since $\boldsymbol{v}^{(0)} \perp \boldsymbol{r}^{(0)}$ at the origin and also perpendicular to $\boldsymbol{B}_0$) and therefore gives no contribution to $\boldsymbol{F}$, while $B_r\hat{\mathbf{e}}_r$ contributes to $\boldsymbol{F}_\parallel$ and $B_z\hat{\mathbf{e}}_z$ contributes to $\boldsymbol{F}_\perp$. Hence, we can split equation (3.19) as follows:

$$\boldsymbol{F}_\parallel = q(\boldsymbol{v}^{(0)} \times \hat{\mathbf{e}}_r) r^{(0)} \frac{\partial B_r}{\partial r} = |q| v^{(0)} r^{(0)} \frac{\partial B_r}{\partial r} \hat{\mathbf{e}}_z \tag{3.21}$$

$$\boldsymbol{F}_\perp = q(\boldsymbol{v}^{(0)} \times \hat{\mathbf{e}}_z) r^{(0)} \frac{\partial B_z}{\partial r} = -|q| v^{(0)} r^{(0)} \frac{\partial B_z}{\partial r} \hat{\mathbf{e}}_r. \tag{3.22}$$

Note that if $q > 0$ we have $\boldsymbol{v}^{(0)} \times \hat{\mathbf{e}}_r = v^{(0)}\hat{\mathbf{e}}_z$, whereas if $q < 0$ we have $\boldsymbol{v}^{(0)} \times \hat{\mathbf{e}}_r = -v^{(0)}\hat{\mathbf{e}}_z$. We rewrite the prefactor in the previous equations as

$$|q| v^{(0)} r^{(0)} = 2\frac{\frac{1}{2} m v^{(0)2}}{2B_0} = 2\frac{W_\perp}{B_0}, \tag{3.23}$$

where $W_\perp$ is the perpendicular component of the kinetic energy.

The magnetic moment $\boldsymbol{\mu}$

The circular motion of the particles around $\boldsymbol{B}_0$ create a ring current which also generates a magnetic field $\boldsymbol{B}_i$ oriented antiparallel to the external magnetic field (diamagnetic properties of a plasma). The magnetic moment $\boldsymbol{\mu}$ associated with the circulating current is normal to the area bounded by the particle orbit and is given by

$$|\boldsymbol{\mu}| = I \cdot A. \tag{3.24}$$

This circulating current corresponds to a flow of charge and is given by

$$I = \frac{|q|}{T_c} = \frac{|q|}{2\pi}\omega_c. \tag{3.25}$$

With $A = \pi r_c^2$ and $r_c = \frac{v_\perp}{\omega_c}$ and $\omega_c = \frac{|q|B}{m}$ the magnitude $|\boldsymbol{\mu}|$ becomes

$$|\mu| = \frac{1}{2}|q|\omega_c r_c^2 = \frac{\frac{1}{2} m v_\perp^2}{2B} = \frac{W_\perp}{B}. \tag{3.26}$$

The magnetic moment points into opposite direction of the magnetic field, therefore

$$\boldsymbol{\mu} = -\frac{W_\perp}{B^2}\boldsymbol{B}. \tag{3.27}$$

Now we can write (3.21) and (3.22) as

$$\boldsymbol{F}_\parallel = 2|\boldsymbol{\mu}|\frac{\partial B_r}{\partial r}\hat{\mathbf{e}}_z, \qquad \boldsymbol{F}_\perp = -2|\boldsymbol{\mu}|\frac{\partial B_z}{\partial r}\hat{\mathbf{e}}_r. \tag{3.28}$$

The results apply to positively and negatively charged particles. The average values of $\boldsymbol{F}_\parallel$ and $\boldsymbol{F}_\perp$ over one gyration period are given by

$$\left\langle \boldsymbol{F}_\parallel \right\rangle = 2|\boldsymbol{\mu}|\frac{1}{2\pi}\oint \frac{\partial B_r}{\partial r}\hat{\mathbf{e}}_z \,\mathrm{d}\varphi = 2|\boldsymbol{\mu}|\hat{\mathbf{e}}_z \left\langle \frac{\partial B_r}{\partial r} \right\rangle \tag{3.29}$$

$$\langle \boldsymbol{F}_\perp \rangle = -2|\boldsymbol{\mu}|\frac{1}{2\pi}\oint \frac{\partial B_z}{\partial r}\hat{\mathbf{e}}_r \,\mathrm{d}\varphi = 2|\boldsymbol{\mu}| \left\langle \hat{\mathbf{e}}_r \frac{\partial B_z}{\partial r} \right\rangle. \tag{3.30}$$

$\left\langle \boldsymbol{F}_\parallel \right\rangle$ leads to an acceleration of the guiding centre along $\boldsymbol{B}$ due to a radial variation of the radial component of $\boldsymbol{B}$ or due to *divergence* terms of $\boldsymbol{B}$.

$\langle \boldsymbol{F}_\perp \rangle$ leads to a transverse drift of the guiding centre due to *gradient* terms of $\boldsymbol{B}$.

Parallel force

We now proceed to evaluate each force term separately. Note that from $\vec{\nabla} \cdot \boldsymbol{B} = 0$ we have, in cylindrical coordinates,

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(rB_r)}_{=\frac{\partial B_r}{\partial r}+\frac{B_r}{r}} + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(B_\varphi\right) + \frac{\partial}{\partial z}(B_z) = 0. \tag{3.31}$$

Since at $r = 0$ we have $B_r = 0$ and B_r changes only very slightly with r we find

$$\frac{\partial B_r}{\partial r} = \frac{B_r}{r} \quad \Rightarrow \quad \frac{\partial B_r}{\partial r} = -\frac{1}{2}\left(\frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z}\right). \tag{3.32}$$

Hence, taking the average over one gyration period

$$\left\langle \frac{\partial B_r}{\partial r} \right\rangle = -\frac{1}{2r}\left\langle \frac{\partial B_\varphi}{\partial \varphi} \right\rangle - \frac{1}{2}\left\langle \frac{\partial B_z}{\partial z} \right\rangle. \tag{3.33}$$

We start by looking at the first term of (3.33). Since $\boldsymbol{B}$ is single valued, moving along the particle orbit B_φ may vary but it has to have the same value at the end again, thus the integral is zero

$$\frac{1}{r}\left\langle \frac{\partial B_\varphi}{\partial \varphi} \right\rangle = \frac{1}{2\pi r}\oint \frac{\partial B_\varphi}{\partial \varphi}\,\mathrm{d}\varphi = 0. \tag{3.34}$$

Now we consider the second term of (3.33). We note that $\frac{\partial B_z}{\partial z}$ is a very slowly varying function inside the particle orbit, thus it can be taken out of the integral

$$\left\langle \frac{\partial B_z}{\partial z} \right\rangle = \frac{1}{2\pi}\oint \frac{\partial B_z}{\partial z}\,\mathrm{d}\varphi = \frac{\partial B_z}{\partial z} \approx \frac{\partial B}{\partial z}. \tag{3.35}$$

It is justifiable to replace B_z by B since all spatial variations are very small. Finally we obtain from (3.34), (3.35) and (3.33)

$$\left\langle \frac{\partial B_r}{\partial r} \right\rangle = -\frac{1}{2}\frac{\partial B}{\partial z}. \tag{3.36}$$

Using this the parallel force becomes

$$\langle \boldsymbol{F}_\parallel \rangle = -|\boldsymbol{\mu}|\frac{\partial B}{\partial z}\hat{\mathbf{e}}_z = -|\boldsymbol{\mu}|(\vec{\nabla}B)_\parallel. \tag{3.37}$$

Perpendicular force

For the perpendicular force it is convenient to consider a two-dimensional cartesian coordinate system with $x = r\cos\varphi$ and $y = r\sin\varphi$. Then

$$\frac{\partial}{\partial r} = \frac{\mathrm{d}x}{\mathrm{d}r}\frac{\partial}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}r}\frac{\partial}{\partial y} = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}. \tag{3.38}$$

Therefore we obtain

$$\left\langle \hat{\mathbf{e}}_r \frac{\partial B_z}{\partial r} \right\rangle = \left\langle \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \left(\cos\varphi \frac{\partial B_z}{\partial x} + \sin\varphi \frac{\partial B_z}{\partial y} \right) \right\rangle$$
$$= \left\langle \cos^2\varphi \frac{\partial B_z}{\partial x} \hat{\mathbf{e}}_x \right\rangle + \left\langle \sin\varphi\cos\varphi \frac{\partial B_z}{\partial x} \hat{\mathbf{e}}_y \right\rangle + \left\langle \cos\varphi\sin\varphi \frac{\partial B_z}{\partial y} \hat{\mathbf{e}}_x \right\rangle$$
$$+ \left\langle \cos^2\varphi \frac{\partial B_z}{\partial y} \hat{\mathbf{e}}_y \right\rangle. \tag{3.39}$$

Nest we approximate $\frac{\partial B_z}{\partial x} = \frac{\partial B}{\partial x}$ and $\frac{\partial B_z}{\partial y} = \frac{\partial B}{\partial y}$. Since these terms are slowly varying functions inside the particle orbit they can be taken outside the integral sign. Noting that $\langle \sin\varphi\cos\varphi \rangle = 0$ and $\langle \cos^2\varphi \rangle = \langle \sin^2\varphi \rangle = \frac{1}{2}$, we obtain

$$\left\langle \hat{\mathbf{e}}_r \frac{\partial B_z}{\partial r} \right\rangle = \frac{1}{2}\frac{\partial B}{\partial x}\hat{\mathbf{e}}_x + \frac{1}{2}\frac{\partial B}{\partial y}\hat{\mathbf{e}}_y. \tag{3.40}$$

Substituting this into the expression of the perpendicular force (3.22) this results in

$$\langle \boldsymbol{F}_\perp \rangle = -|\boldsymbol{\mu}| \left(\frac{\partial B}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial B}{\partial y}\hat{\mathbf{e}}_y \right) = -|\boldsymbol{\mu}| (\vec{\nabla} B)_\perp. \tag{3.41}$$

Total average force

We can now proceed to write down a general expression for the *total average force* as

$$\langle \boldsymbol{F} \rangle = \langle \boldsymbol{F}_\parallel \rangle + \langle \boldsymbol{F}_\perp \rangle = -|\boldsymbol{\mu}|(\vec{\nabla} B)_\parallel - |\boldsymbol{\mu}|(\vec{\nabla} B)_\perp = -|\boldsymbol{\mu}|(\vec{\nabla} B). \tag{3.42}$$

3.5 Gradient drift

Since $\langle \boldsymbol{F}_\perp \rangle$ is perpendicular to the magnetic field it causes the guiding centre to drift with the velocity (c. f. $\boldsymbol{E} \times \boldsymbol{B}$ drift)

$$\boldsymbol{v}_G = \frac{\langle \boldsymbol{F}_\perp \rangle \times \boldsymbol{B}}{qB^2} = -\frac{|\boldsymbol{\mu}|}{q} \frac{\vec{\nabla} B \times \boldsymbol{B}}{B^2}. \tag{3.43}$$

This gradient drift is perpendicular to $\boldsymbol{B}$ and the field gradient. Its directions depends on the charge sign. Thus, positive and negative charges drift in opposite directions.

The physical reason for this gradient drift can be seen as follows. Since the LARMOR radius of the particle orbit decreases as the magnetic field increases, the radius of curvature of the orbit is smaller in the regions of stronger $\boldsymbol{B}$ field. The positive ions gyrate in the clockwise direction for $\boldsymbol{B}$ pointing towards the observer, while the electrons gyrate in the counter clockwise directions, so that positive ions drift to the left and electrons to the right (see figure 6).

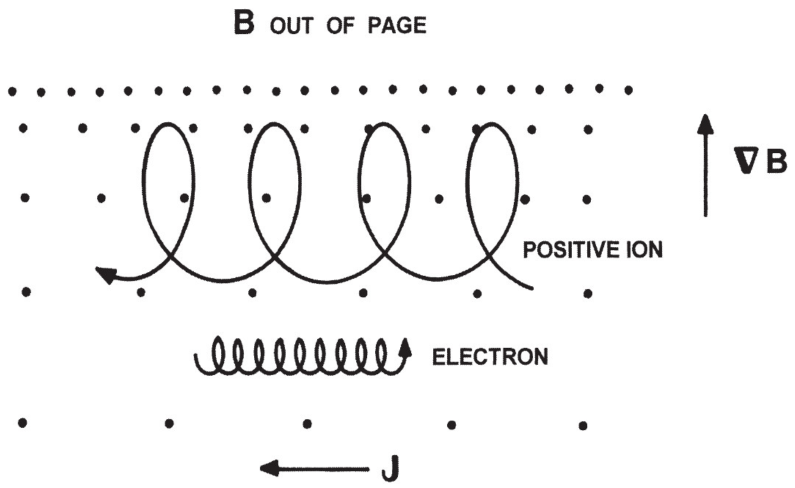


Fig. 6: Charged particles drift to to a field gradient ($\vec{\nabla} B$) perpendicular to $\boldsymbol{B}$.

3.6 Parallel acceleration of guiding centre

The expression (3.37) for the parallel force $\langle \boldsymbol{F}_{\parallel} \rangle$ shows that, when the magnetic field has a longitudinal variation, i. e. divergence of the field lines along the z-direction, an axial force along z accelerates the particle in the direction towards decreasing field strength, irrespective of whether the particle is positively or negatively charged (see figure 7).

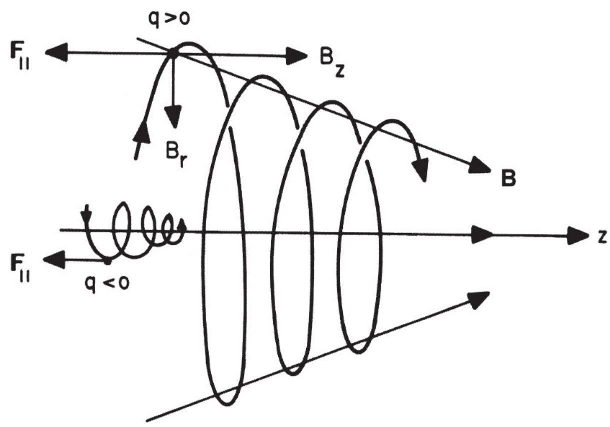


Fig. 7: Repulsion of gyrating charges from a region of converging magnetic field lines.

There are several important consequences of this repulsion of gyrating charges from a region of converging field lines.

3.6.1 Invariance of magnetic moment and flux

Using (3.37), the parallel force is given as

$$m\frac{\mathrm{d}v_{\parallel}}{\mathrm{d}t}\hat{\mathbf{e}}_z = -|\boldsymbol{\mu}|\frac{\partial B}{\partial z}\hat{\mathbf{e}}_z \quad \text{with} \quad |\boldsymbol{\mu}| = \frac{\frac{1}{2}mv_{\perp}^2}{B}. \tag{3.44}$$

If we multiply both sides of this equation by $v_\parallel = \frac{\mathrm{d}z}{\mathrm{d}t}$ we obtain

$$m v_\parallel \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} m v_\parallel^2\right) = -\frac{W_\perp}{B}\frac{\partial B}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}. \tag{3.45}$$

Since the total kinetic energy of a charged particle is constant in magnetostatic fields $W_\perp + W_\parallel$ =const., if follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(W_\perp) &= -\frac{\mathrm{d}}{\mathrm{d}t}(W_\parallel) = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2} m v_\parallel^2\right)\\ &\overset{(3.45)}{=} \frac{W_\perp}{B}\left(\frac{\partial B}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right) = \frac{W_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t},\end{aligned} \tag{3.46}$$

where $\frac{\mathrm{d}B}{\mathrm{d}t}$ represents the rate of change of B as seen by the particle as it moves in the spatially varying magnetic field. Comparing this result with the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}(W_\perp) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W_\perp B}{B}\right) = \frac{W_\perp}{B}\frac{\mathrm{d}B}{\mathrm{d}t} + B\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{W_\perp}{B}\right)}_{=0}. \tag{3.47}$$

Thus we conclude

First adiabatic invariant

$$|\boldsymbol{\mu}| = \frac{W_\perp}{B} = \text{const.} \tag{3.48}$$

Therefore, as the particle moves into regions of converging $\boldsymbol{B}$, the LARMOR frequency and perpendicular velocity increase,

$$\omega_c = \frac{|q|}{m}B, \qquad v_\perp = \sqrt{\frac{2B|\boldsymbol{\mu}|}{m}}, \qquad r_c = \frac{1}{|q|}\sqrt{\frac{2m|\boldsymbol{\mu}|}{B}}m, \tag{3.49}$$

while the cyclotron radius decreases. However, the magnetic moment remains constant. This is only valid for slightly inhomogeneous magnetic fields, when the spatial variation of $\boldsymbol{B}$ inside the particle orbit is small. Consequently the orbital magnetic moment is said to be an *adiabatic invariant*.

The *magnetic flux* Φ_m enclosed by one orbit of the particle is given by

$$\Phi_m = \int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \pi r_c^2 B = \pi\frac{m^2 v_\perp^2}{q^2 B} = \frac{2\pi m}{q^2}|\boldsymbol{\mu}| = \text{const.}, \tag{3.50}$$

hence, as the charged particle moves in a region of converging $\boldsymbol{B}$ field, it will orbit with increasingly smaller radius, so that the magnetic flux enclosed by the orbit remains constant.

3.6.2 Magnetic mirror effect

As a consequence of the adiabatic invariance of $|\boldsymbol{\mu}|$ and Φ_m as the particle moves into a region of converging magnetic field lines its transverse kinetic energy $W_\perp$ increases, while its parallel kinetic energy $W_{\parallel}$ decreases in order to keep $|\boldsymbol{\mu}|$ and the total energy constant. Ultimately, if the magnetic field becomes strong enough, the particle velocity in the direction of increasing field may eventually come to zero and then be reversed. After reversion, the particle is accelerated towards decreasing magnetic field, while its transverse velocity diminishes. Thus, the particle is *reflected* from the region of converging field lines. This phenomenon is called the *magnetic mirror effect* and is the basis for one of the primary schemes of plasma confinement.

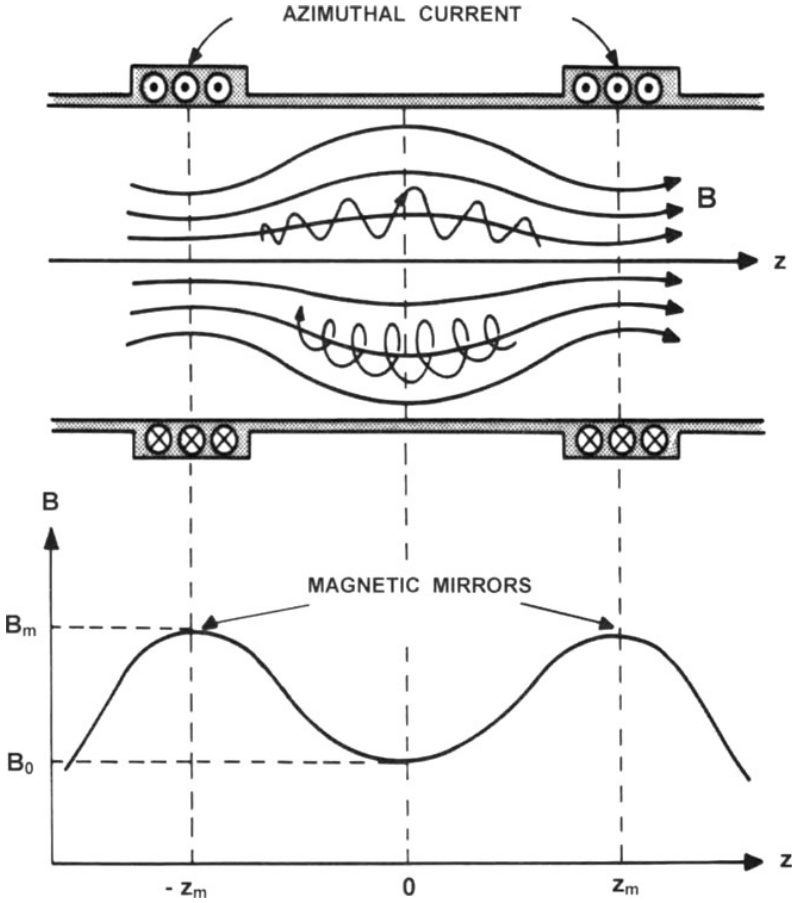


Fig. 8: Schematic diagram showing the arrangement of coils to produce two coaxial magnetic mirrors facing each other, for plasma confinement, and the relative intensity variation of the magnetic field.

When we consider two coaxial magnetic mirrors as illustrated in figure 8, the charged particles may be reflected by the magnetic mirrors and travel back and forth in the space between them, becoming trapped. This trapping region is called a *magnetic bottle*. However, the trapping in a magnetic mirror system is not perfect. The effectiveness of this coaxial mirror system is given by the *mirror ratio*

$$\frac{B_m}{B_0} \quad \text{with} \quad B_m = B(z = \pm z_m), B_0 = B(z = 0), \tag{3.51}$$

where B_m is the magnetic field density at the point of reflection, where the *pitch angle* of the particle is $\frac{\pi}{2}$.

Now consider a charged particle having a pitch angle α_0 at the centre of the magnetic bottle having a total speed v. The constancy of the magnetic moment leads to

$$\frac{W_\perp}{B} = \frac{\frac{1}{2}mv^2 \sin^2\alpha}{B} = \frac{\frac{1}{2}mv^2 \sin^2\alpha_0}{B_0} \tag{3.52}$$

where α is the particle pitch angle at the position where the magnetic field is B. Thus, at any point inside the magnetic bottle we have

$$\frac{\sin^2\alpha(z)}{B(z)} = \frac{\sin^2\alpha_0}{B_0}. \tag{3.53}$$

Suppose now that this particle is reflected at the bottle neck, then $\alpha = \frac{\pi}{2}$ and

$$\frac{\sin^2\alpha_0}{B_0} = \frac{1}{B_m} \quad \Rightarrow \quad \alpha_0 = \arcsin\left(\sqrt{\frac{B_0}{B_m}}\right) = \arcsin\left(\frac{v_\perp}{v}\right)_0. \tag{3.54}$$

Therefore we observe that for a magnetic bottle with fixed mirror ratio B_m/B_0 reflections will only take place for particles with $\alpha_0 \geq \alpha_{0,\min}$ reflection. If the pitch angle of the particle at the centre is less than α_0, it will not be reflected and escapes through the ends of the mirror system.

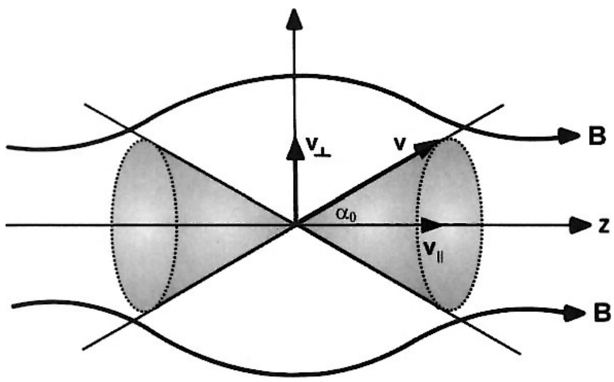


Fig. 9: The loss cone in a coaxial magnetic mirror system. Particles with an initial pitch angle $\alpha_0 < \alpha_{0,\min}$ will escape from the magnetic bottle.

Therefore there is a *loss cone* with half-angle α_0 as shown in figure 9 where particles that have velocity vectors with a pitch angle falling inside the loss cone are not trapped. Devices that have no end, with geometries such that the magnetic field lines close on themselves offer many advantages for plasma confinement. Toroidal geometries have no end, but it turns out that confinement of a plasma inside a toroidal magnetic field does not provide a plasma equilibrium situation due to the inhomogeneity of the field. In this case a poloidal magnetic field is normally superposed on the toroidal field, resulting in *helical* field lines (as in the *Tokamak*). A major problem in most plasma confinement schemes is that instabilities and small fluctuations from the equilibrium configuration are always present leading to a rapid escape of particles.

An example of a natural magnetic bottle is the Earth's magnetic field, which traps charged particles of solar and cosmic origin. These charged particles trapped in the Earth's magnetic field constitute the so-called *Van Allen radiation belts*. The electrons and protons that are trapped in these belts spiral in almost helical paths along the field lines and towards the magnetic poles, where they are eventually reflected. The particles bounce back and forth between the poles. Additionally they are also subject to a gradient drift and a curvature drift in the east-west direction which will be discussed in the next section.

3.7 Curvature drift

So far we have neglected effects associated with the curvature of the magnetic field. As stated before a $\boldsymbol{B}$-field with only curvature terms does not satisfy $\vec{\nabla} \times \boldsymbol{B} = 0$, so that in practice the gradient and the curvature drifts will always be present simultaneously. However, both drifts are to first order approximation independent and can be discussed separately.

We investigate the effects of $\frac{\partial B_x}{\partial z}$ and $\frac{\partial B_y}{\partial z}$ on the motion of a small particle. We assume that these terms are so small that the radius of curvature of the magnetic field lines is much larger than the cyclotron radius.

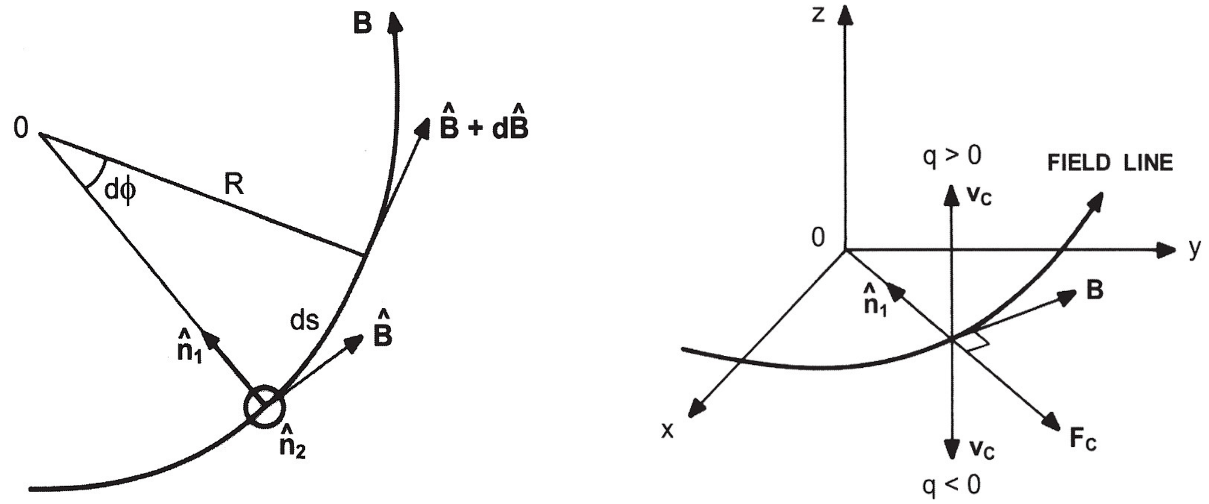


Fig. 10: Left: Curved magnetic field line showing the unit vector $\hat{\mathbf{B}}$ along the field line, the principal normal $\hat{\mathbf{n}}_1$ and the binormal $\hat{\mathbf{n}}_2$ at any arbitrary point. The local radius of curvature is R.
Right: Relative direction of the particles guiding centre drift velocity $\boldsymbol{v}_c$ due to the curvature of the magnetic field lines.

Let us introduce a local coordinate system gliding along the magnetic field line with the particles longitudinal velocity $\boldsymbol{v}_\parallel$. Since this is not an inertial system, a centrifugal force will be present. This local coordinate system is specified by the orthogonal set of unit vectors $\hat{\mathbf{B}}, \hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$, where $\hat{\mathbf{B}}$ is along the field line, $\hat{\mathbf{n}}_1$ is along the principal normal to the field line and $\hat{\mathbf{n}}_2$ is along the binormal to the curved magnetic field liene, as indicated in figure 10 (left).

The centrifugal force $\boldsymbol{F}_c$ acting on the particle seen from this non-inertial system is given by

$$\boldsymbol{F}_c = -\frac{m v_\parallel^2}{R}\hat{\mathbf{n}}_1 \tag{3.55}$$

where R denotes the local radius of curvature of the magnetic field line. From equation (2.23) the curvature drift associated with this force is

$$\boldsymbol{v}_c = \frac{\boldsymbol{F}_c \times \boldsymbol{B}}{qB^2} = -\frac{m v_\parallel^2}{R\, q\, B^2}(\hat{\mathbf{n}}_1 \times \boldsymbol{B}). \tag{3.56}$$

In order to express the unit vector $\hat{\mathbf{n}}_1$ in terms of the unit vector $\hat{\mathbf{B}}$ along the magnetic field line, lets write the element of arc $\mathrm{d}s$ along the field line as $\mathrm{d}s = R\,\mathrm{d}\phi$. If $\mathrm{d}\hat{\mathbf{B}}$ denotes the change in $\boldsymbol{B}$ due to the displacement $\mathrm{d}s$, then $\mathrm{d}\hat{\mathbf{B}} \sim \hat{\mathbf{n}}_1$ and its magnitude is $|\mathrm{d}\hat{\mathbf{B}}| = |\hat{\mathbf{B}}|\,\mathrm{d}\phi = \mathrm{d}\phi$. Consequently

$$\mathrm{d}\hat{\mathbf{B}} = \hat{\mathbf{n}}_1\,\mathrm{d}\phi \quad \Rightarrow \frac{\mathrm{d}\hat{\mathbf{B}}}{\mathrm{d}s} = (\hat{\mathbf{B}}\cdot\vec{\nabla})\hat{\mathbf{B}} = \frac{\hat{\mathbf{n}}_1}{R}. \tag{3.57}$$

Incorporating this result into equation (3.55) we obtain

$$\boldsymbol{F}_c = -m v_\parallel^2 (\hat{\mathbf{B}}\cdot\vec{\nabla})\hat{\mathbf{B}} = -\frac{2W_\parallel}{B^2}[(\hat{\mathbf{B}}\cdot\vec{\nabla})\hat{\mathbf{B}}]_\perp. \tag{3.58}$$

This force is perpendicular to the magnetic field $\boldsymbol{B}$, since it is in the $-\hat{\mathbf{n}}_1$ direction and gives rise to a curvature drift velocity

$$\boldsymbol{v}_C = -\frac{m v_\parallel^2}{qB^2}[(\hat{\mathbf{B}}\cdot\vec{\nabla})\hat{\mathbf{B}}] \times \boldsymbol{B} = -\frac{2W_\parallel}{qB^4}[(\hat{\mathbf{B}}\cdot\vec{\nabla})\hat{\mathbf{B}}] \times \boldsymbol{B}. \tag{3.59}$$

Thus, at each point, the curvature drift is perpendicular to the plane of the magnetic field line as shown in figure 10 (right).

3.8 Combined gradient-curvature drift

As stated before curvature and gradient drift always appear together and both point in the same direction, since $\vec{\nabla}B$ points in opposite direction to $\boldsymbol{F}_c$. These two drifts can be added up to form the combined gradient-curvature drift. Thus, using (3.43) and (3.59) we find

$$\boldsymbol{v}_{GC} = \boldsymbol{v}_G + \boldsymbol{v}_C = -\frac{W_\perp}{qB^3}(\vec{\nabla}B) \times \boldsymbol{B} - \frac{2W_\parallel}{qB^4}[(\hat{\mathbf{B}}\cdot\vec{\nabla})\hat{\mathbf{B}}] \times \boldsymbol{B}. \tag{3.60}$$

When volume currents are not present so that $\vec{\nabla} \times \boldsymbol{B} = 0$, we can use $\vec{\nabla}\left(\frac{1}{2}B^2\right) = B(\vec{\nabla}B)$ to write the gradient-curvature drift in a compact form

$$\boldsymbol{v}_{GC} = -\frac{1}{qB^4}(2W_\parallel^2 + W_\perp^2)\vec{\nabla}\left(\frac{1}{2}B^2\right) \times \boldsymbol{B}. \tag{3.61}$$

In the Earth's magnetosphere, near the equatorial plane, both drifts cause the positively charged particles to slowly drift westward and the negative ones eastward, resulting in an east to west current, known as the *ring current.*

4 Basic plasma phenomena

4.1 Electron plasma oscillations

One fundamental plasma property is its tendency to maintain electric charge neutrality on macroscopic scales under equilibrium conditions. When this charge neutrality is disturbed by a temporarily imbalance of charge, large COULOMB forces come into play, which tend to restore macroscopic charge neutrality. Since these forces cannot be naturally sustained, high frequency plasma oscillations are excited, which enable the plasma to maintain its average electrical neutrality.

As an example consider a small spherical region inside the plasma and suppose that a perturbation in the form of excess of negative charge is introduced in this small region. The corresponding electric field is radial and points towards the center, forcing the electrons to move outward. Due to their inertia, the electrons move further than necessary to resume the state of electrical neutrality. This creates an excess of positive charges causing the electrons to move inward again. This sequence of outward and inward electron movement results in electron plasma oscillations. Since the ions (due to their higher mass) are unable to follow the rapidity of the electron oscillations, their motion is often neglected.

Now we study the characteristics of the electron plasma oscillations using the *cold plasma model* in which the particle thermal motion and pressure gradient forces are not taken into account. We neglect the ion motion and assume a small electron density perturbation such that

$$n_e(\boldsymbol{r},t) = n_0 + n_e'(\boldsymbol{r},t), \tag{4.1}$$

where n_0 is a constant number density and $|n_e'| \ll n_0$. Similarly we assume that $\boldsymbol{E}(\boldsymbol{r},t)$ and the average electron velocity $\boldsymbol{u}_e(\boldsymbol{r},t)$ are first-order perturbations. We can derive the *plasma frequency* ω_p using the linearized *continuity* and *momentum* equations

$$\frac{\partial}{\partial t} n_e'(\boldsymbol{r},t) + n_0 \vec{\nabla} \cdot \boldsymbol{u}_e(\boldsymbol{r},t) = 0 \tag{4.2}$$

$$\frac{\partial}{\partial t} \boldsymbol{u}_e(\boldsymbol{r},t) = -\frac{e}{m_e} \boldsymbol{E}(\boldsymbol{r},t). \tag{4.3}$$

Considering singly charged ions the total charge density is simply given by the perturbation value of the electron number density n_e' if we assume constant and uniform ions. Therefore, GAUSS law states

$$\vec{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r},t) = \frac{\varrho(\boldsymbol{r},t)}{\varepsilon_0} = -\frac{e}{\varepsilon_0} n_e'(\boldsymbol{r},t). \tag{4.4}$$

We now try to solve this set of equations for the electron number density by taking the divergence of (4.3) and using (4.2) to substitute for $\vec{\nabla} \cdot \boldsymbol{u}_e$, we obtain

$$\frac{\partial^2}{\partial t^2} n_e'(\boldsymbol{r},t) - \frac{e n_0}{m_e} \vec{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r},t) = 0$$

$$\overset{(4.4)}{\Rightarrow} \frac{\partial^2}{\partial t^2} n_e'(\boldsymbol{r},t) + \omega_p^2 n_e'(\boldsymbol{r},t) = 0, \quad \text{where} \quad \omega_p = \sqrt{\frac{n_0 e^2}{m_e \varepsilon_0}} \tag{4.5}$$

is the *electron plasma frequency* we already introduced in equation (1.8). We observe that $n'_e(\boldsymbol{r}, t)$ varies harmonically in time leading to the general solution

$$n'_e(\boldsymbol{r}, t) = n'_e(\boldsymbol{r}) \exp\bigl(-\mathrm{i}\omega_p t\bigr). \tag{4.6}$$

In fact, all first-order perturbations have a harmonic time variation at the plasma frequency. To justify this statement its convenient to start with the assumption that all first-order quantities vary harmonically in time with $\exp(-\mathrm{i}\omega t)$. Equations (4.2) and (4.3) become, in this case

$$n'_e = -\frac{\mathrm{i}}{\omega} n_0 \vec{\nabla} \cdot \boldsymbol{u}_e, \qquad \boldsymbol{u}_e = -\frac{\mathrm{i}e}{\omega m_e} \boldsymbol{E} \tag{4.7}$$

$$= -\frac{n_0 e}{\omega^2 m_e} \vec{\nabla} \cdot \boldsymbol{E}. \tag{4.8}$$

Substituting this expression into equation (4.4) yields

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right) \vec{\nabla} \cdot \boldsymbol{E} = 0, \tag{4.9}$$

which shows that nontrivial solutions require $\omega = \omega_p$. Therefore, all the perturbations vary harmonically in time at the electron plasma frequency. Further, for all variables there is no change in phase from point to point implying the absence of wave propagation. The oscillations are therefore *stationary*. Also $\boldsymbol{u}_e \parallel \boldsymbol{E}$ shows that the electron velocity is in the same direction as the electric field, so that these oscillations are *longitudinal*.

The electron plasma oscillations are also *electrostatic* in character. For this we consider FARADAY's and AMPÈRE's law with harmonic time variation

$$\vec{\nabla} \times \boldsymbol{E} = \mathrm{i}\omega \boldsymbol{B} \tag{4.10}$$

$$\vec{\nabla} \times \boldsymbol{B} = \mu(\boldsymbol{j} - \mathrm{i}\omega\varepsilon_0 \boldsymbol{E}). \tag{4.11}$$

where the electric current density is given by

$$\boldsymbol{j} = -en_0 \boldsymbol{u}_e \overset{(4.7)}{=} \frac{\mathrm{i} n_0 e^2}{\omega m_e} \boldsymbol{E} \quad \Rightarrow \vec{\nabla} \times \boldsymbol{B} = -\frac{\mathrm{i}\omega}{c^2} \varepsilon_r \boldsymbol{E} \tag{4.12}$$

where we have defined a relative permittivity by

$$\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2}. \tag{4.13}$$

For the electron plasma oscillations we have $\omega = \omega_p$ so that $\varepsilon_r = 0$ and (4.12) is just $\vec{\nabla} \times \boldsymbol{B} = 0$. Since the curl of the gradient of any scalar function vanishes we may write

$$\boldsymbol{B} = \vec{\nabla}\psi \quad \Rightarrow \vec{\nabla} \cdot \left(\vec{\nabla}\psi\right) = \Delta\psi = 0, \tag{4.14}$$

because $\vec{\nabla} \cdot \boldsymbol{B} = 0$. The only solution of this equation, which is not singular and finite at $\boldsymbol{r}| < \infty$ is ψ = constant, so that $\boldsymbol{B} = 0$. Hence there is no magnetic field associated with these space charge oscillations.

In summary, the electron plasma oscillations are stationary, longitudinal and electrostatic. They are also referred to as *Langmuir oscillations.* When the effect of pressure gradient forces is included in the equation of motion (4.3), these oscillations become propagating disturbances, commonly known as *Langmuir waves.*

4.2 Debye shielding

To examine the mechanism by which the plasma strives to shield its interior from a disturbing electric field consider a plasma whose equilibrium state is perturbed by an electric field due to an external charged particle. We assume this *test particle* to have a positive charge $+Q$ and choose a spherical coordinate system whose origin coincides with the position of the test particle. We want to determine the electrostatic potential $\Phi(\boldsymbol{r})$ near the test charge due to combined effects of the test charge and the distribution of charged particles surrounding it. The number densities of electrons $n_e(\boldsymbol{r})$ and of the ions $n_i(\boldsymbol{r})$ will be slightly different near the origin, whereas at large distances we have $n_e(\infty) = n_i(\infty) = n_0$. This is a steady-state problem under the action of a conservative electric field

$$\boldsymbol{E}(\boldsymbol{r}) = -\vec{\boldsymbol{\nabla}}\Phi(\boldsymbol{r}). \tag{4.15}$$

Assuming that electrons and ions have the same temperature their distributions are described by

$$n_e(\boldsymbol{r}) = n_0 \exp\left(\frac{e\Phi(\boldsymbol{r})}{k_B T}\right), \qquad n_i(\boldsymbol{r}) = n_0 \exp\left(-\frac{e\Phi(\boldsymbol{r})}{k_B T}\right). \tag{4.16}$$

The total electric charge density $\varrho(\boldsymbol{r})$ including the test charge Q can be expressed as

$$\begin{aligned} \varrho(\boldsymbol{r}) &= -e[n_e(\boldsymbol{r}) - n_i(\boldsymbol{r})] + Q\delta(\boldsymbol{r}) \\ &= -en_0\left[\exp\left(\frac{e\Phi(\boldsymbol{r})}{k_B T}\right) - \exp\left(-\frac{e\Phi(\boldsymbol{r})}{k_B T}\right)\right] + Q\delta(\boldsymbol{r}). \end{aligned} \tag{4.17}$$

Using POISSON's equation $\varepsilon_0 \boldsymbol{\nabla}^2\Phi(\boldsymbol{r}) = -\varrho(\boldsymbol{r})$ we find

$$\boldsymbol{\nabla}^2\Phi(\boldsymbol{r}) - \frac{en_0}{\varepsilon_0}\left[\exp\left(\frac{e\Phi(\boldsymbol{r})}{k_B T}\right) - \exp\left(-\frac{e\Phi(\boldsymbol{r})}{k_B T}\right)\right] = -\frac{Q}{\varepsilon_0}\delta(\boldsymbol{r}) \tag{4.18}$$

which allows the evaluation of the electrostatic potential. In order to proceed analytically, we assume now that the perturbing electrostatic potential is weak so that the electrostatic energy is much smaller than the mean thermal energy

$$e\Phi(\boldsymbol{r}) \ll k_B T. \tag{4.19}$$

Under this condition we can expand the exponentials into a first-order TAYLOR series expansion which simplifies (4.18) to

$$\boldsymbol{\nabla}^2\Phi(\boldsymbol{r}) - \frac{2}{\lambda_D^2}\Phi(\boldsymbol{r}) = -\frac{Q}{\varepsilon_0}\delta(\boldsymbol{r}) \tag{4.20}$$

where λ_D denotes the DEBYE length

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{n_0 e^2}} = \frac{1}{\omega_p}\sqrt{\frac{k_B T}{m_e}} \sim \frac{v_{\text{th}}}{\omega_p}. \tag{4.21}$$

The product of plasma frequency and DEBYE length is proportional to the thermal velocity v_{th}. Since the problem has spherical symmetry, the electrostatic potential depends only on the radial distance r measure from the position of the test particle. Thus, using spherical coordinates we can write (4.20) (for $r \neq 0$) as

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^2\frac{\mathrm{d}}{\mathrm{d}r}\Phi(\boldsymbol{r})\right] - \frac{2}{\lambda_D^2}\Phi(\boldsymbol{r}) = 0 \qquad (r \neq 0). \tag{4.22}$$

For an intuitive solution we note that for an isolated particle in free space, the electric potential is simply

$$\Phi_c(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \Rightarrow \quad \Phi(r) = \Phi_c(r)F(r) = \frac{Q}{4\pi\varepsilon_0}\frac{F(r)}{r}. \tag{4.23}$$

In the very close proximity of the test particle the electrostatic potential should be the same as in free space. Hence we may modify the solution with a function $\lim_{r\to 0} F(r) = 1$. Furthermore, the electrostatic potential has to vanish at infinity. Substituting (4.23) into (4.22) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}F(r) = \frac{2}{\lambda_D^2}F(r) \quad \Rightarrow F(r) = A\exp\left(-\frac{\sqrt{2}r}{\lambda_D}\right). \tag{4.24}$$

The minus sign in the exponent was chosen such that $\Phi(r)$ vanishes for large distances r. Therefore the solution of (4.22) is

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\exp\left(-\frac{\sqrt{2}r}{\lambda_D}\right). \tag{4.25}$$

This result is commonly know as the *Debye potential*, since this non-rigorous derivation was first presented by DEBYE and HUCKEL in their theory of electrolytes. It shows that $\Phi(r)$ becomes much smaller than the ordinary COULOMB potential once r exceeds the *Debye length*. Hence, we can say that charge neutrality is significantly disturbed only over $r \leq \lambda_D$ and for distances larger than λ_D the external charge is effectively shielded (neutralized).

An important point to be noted in the result is that, for $r \to 0$, the DEBYE potential becomes very large and the assumption $e\Phi(r) \ll k_B T$ is unlikely to be fulfilled. To verify the validity of this approximation, note that using (4.25) with $Q = e$ we have

$$\frac{e\Phi}{k_B T} = \frac{e^2}{4\pi\varepsilon_0 r k_B T}\exp\left(-\frac{\sqrt{2}r}{\lambda_D}\right) = \frac{\lambda_D}{3N_D}\frac{1}{r}\exp\left(-\frac{\sqrt{2}r}{\lambda_D}\right) \tag{4.26}$$

where N_D is the number of electrons inside the DEBYE sphere. Since N_D is very large for virtually all plasmas, it is evident that the ration given in (4.26) is much less than one, except when r is less than λ_D/N_D. Therefore, the DEBYE potential is consistent with our approximation if we restrict to distances greater than λ_D/N_D.

Finally we note that in the derivation of the DEBYE potential it is usual to ignore ion motion and assume a constant ion number density. In this case the factor of 2 in equation (4.20) disappears and the expression for the DEBYE potential becomes

$$\Phi(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\exp\left(-\frac{r}{\lambda_D}\right). \tag{4.27}$$

4.3 Electromagnetic waves in a plasma

We begin with a brief review of light waves in vacuum without static magnetic fields (only first order-terms $\boldsymbol{B}$). The relevant MAXWELL equations are

$$\vec{\nabla} \times \boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}, \quad c^2\vec{\nabla} \times \boldsymbol{B} = \frac{\partial}{\partial t}\boldsymbol{E} \tag{4.28}$$

since in a vacuum $\boldsymbol{j} = 0$. Taking the curl of AMPÈRE's law and substituting the time derivative of FARADAY's law we have

$$c^2\vec{\nabla} \times (\vec{\nabla} \times \boldsymbol{B}) = \vec{\nabla} \times \dot{\boldsymbol{E}} = -\ddot{\boldsymbol{B}}. \tag{4.29}$$

Assuming plane waves varying as $\exp(\mathrm{i}(kx - \omega t))$ we have

$$-c^2\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{B}) = -c^2\left[\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{B}) - k^2\boldsymbol{B}\right] = \omega^2\boldsymbol{B}. \tag{4.30}$$

Since $\boldsymbol{k}\cdot\boldsymbol{B} = -\mathrm{i}\vec{\nabla}\cdot\boldsymbol{B} = 0$ the result is

$$\omega^2 = k^2c^2 \quad \text{dispersion relation in vacuum.} \tag{4.31}$$

In a plasma we must add a term $\frac{\boldsymbol{j}}{\varepsilon_0}$ to account for currents due to first-order charged particle motions. The time derivative of AMPÈRE's law then becomes

$$c^2\vec{\nabla} \times \dot{\boldsymbol{B}} = \frac{1}{\varepsilon_0}\frac{\partial \boldsymbol{j}}{\partial t} + \ddot{\boldsymbol{E}} \tag{4.32}$$

while the curl of FARADAY's law is

$$\vec{\nabla} \times \left(\vec{\nabla} \times \boldsymbol{E}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\boldsymbol{E}\right) - \nabla^2\boldsymbol{E} = -\vec{\nabla} \times \dot{\boldsymbol{B}}. \tag{4.33}$$

Eliminating $\vec{\nabla} \times \dot{\boldsymbol{B}}$ by (4.32) and assuming a plane wave dependence results in

$$-\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E}) + k^2\boldsymbol{E} = \frac{\mathrm{i}\omega}{\varepsilon_0c^2}\boldsymbol{j} + \frac{\omega^2}{c^2}\boldsymbol{E}. \tag{4.34}$$

By transverse waves we mean $\boldsymbol{k}\cdot\boldsymbol{E} = 0$, so this becomes

$$(\omega^2 - c^2k^2)\boldsymbol{E} = -\frac{\mathrm{i}\omega}{\varepsilon_0}\boldsymbol{j}. \tag{4.35}$$

If we consider light waves or microwaves, they will be of such high frequency that the ions can be considered as fixed. The current $\boldsymbol{j}$ is then entirely caused by electron motion $\boldsymbol{j} = -n_0e\boldsymbol{v}_e$. From the linearized equation of motion (first-order approximation) we have for a cold plasma

$$m\frac{\partial \boldsymbol{v}_e}{\partial t} = -e\boldsymbol{E} \quad \Rightarrow \quad \boldsymbol{j} = -\frac{n_0e^2}{\mathrm{i}m_e\omega}\boldsymbol{E}. \tag{4.36}$$

Equation (4.35) can now be written as

$$(\omega^2 - c^2k^2)\boldsymbol{E} = \frac{n_0e^2}{\varepsilon_0m_e}\boldsymbol{E} = \omega_p^2\boldsymbol{E}. \tag{4.37}$$

Then the modified dispersion relation for electromagnetic waves propagating in a plasma is

$$\omega^2 = k^2c^2 + \omega_p^2. \tag{4.38}$$

We see that the vacuum relation is modified by a term ω_p^2 reminiscent of plasma oscillations.

Phase and group velocities

The phase velocity of a light wave in a plasma is greater than the velocity of light

$$v_{\mathrm{ph}} = \frac{\omega}{k} = c\frac{\omega}{\sqrt{c^2k^2}} = c\frac{\omega}{\omega^2 - \omega_p^2} = c\frac{1}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} =: \frac{c}{\eta} > c. \tag{4.39}$$

Here we introduced the refractive index of a plasma η as

$$\eta = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \quad \text{refractive index.} \tag{4.40}$$

However, the group velocity v_{gr} cannot exceed the velocity of light. We find

$$v_{\mathrm{gr}} = \frac{\partial\omega}{\partial k} = \frac{kc^2}{\sqrt{\omega_p^2 + k^2c^2}} = \frac{kc^2}{\omega} = c\eta < c. \tag{4.41}$$

For frequencies $\omega < \omega_p$ the refractive index becomes imaginary. This means the electromagnetic wave can no longer propagate in the plasma, it is exponentially damped. Therefore only waves with $\omega > \omega_p$ can propagate in the plasma. For a given ω we can estimate a critical density n_{cr} as

$$n_0 < \frac{\omega^2 \varepsilon_0 m_e}{e^2} = n_{\mathrm{cr}}. \tag{4.42}$$

Electromagnetic waves can only propagate in plasmas with $n_0 < n_{\mathrm{cr}}$. We can express the refractive index in terms of electron densities also as

$$\eta = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \sqrt{1 - \frac{n_e}{n_{\mathrm{cr}}}}. \tag{4.43}$$

5 The pinch effect

In this chapter we present a detailed treatment of plasma confinement for the special case in which the confinement is produced by an azimuthal self-magnetic field, due to an axial current in the plasma in $\hat{\mathbf{e}}_z$ direction.

Consider a cylindrically symmetric plasma with maximum radius R and an axial current density

$$\boldsymbol{j}(r) = -en_e(r)\boldsymbol{u}'_e(r) = j_z(r)\hat{\mathbf{e}}_z \tag{5.1}$$

and a resulting azimuthal magnetic flux density $\boldsymbol{B} = B_\varphi(r)\hat{\mathbf{e}}_\varphi$ as depicted in figure 11.

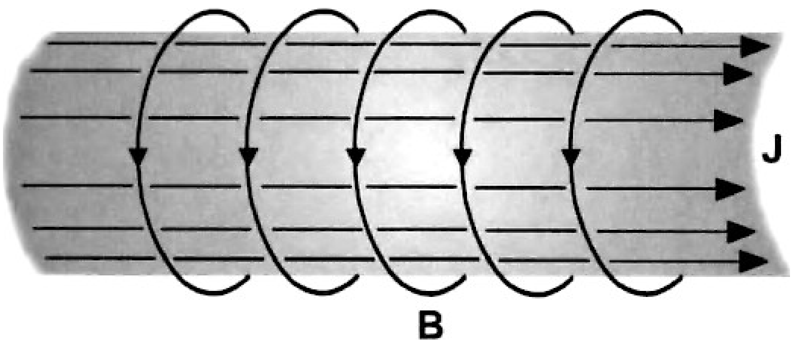


Fig. 11: Pinch configuration in which a magnetoplasma is confined by azimuthal magnetic fields generated by axial currents flowing along the plasma column.

The LORENTZ force $\boldsymbol{j} \times \boldsymbol{B}$ acting on the plasma, forces the column to contract radially. This radial constriction of the plasma column is known as the *pinch effect*. In this case the isobaric surfaces for constant pressure p are concentric cylinders.

As the plasma is compressed, the number density and the temperature increase. The plasmas kinetic pressure

$$p_e(r) = n_e(r)k_B T_e \quad \text{and} \quad p_i(r) = n_i(r)k_B T_i \tag{5.2}$$

counteracts the constriction of the plasma column, whereas the magnetic force acts to confine the plasma. When these counteracting forces are balanced, a steady-state condition establishes in which the plasma is mainly confined the radius R. This situation is referred to as the *equilibrium pinch*. When the column radius changes with time, the situation is known as *dynamic pinch*.

5.1 Equilibrium pinch

For simplicity, the current density, magnetic field and plasma kinetic pressure are assumed to depend only on the distance from the cylinder axis. We do not observe changes with time due to steady-state conditions. We only consider the radial component of the magnetic pressure

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -j_z(r)B_\varphi(r). \tag{5.3}$$

Inside a cylinder of general radius r the total enclosed current $I_z(r)$ is

$$I_z(r) = \int_0^r j_z(r')2\pi r'\,\mathrm{d}r' \quad \Rightarrow \quad \frac{\mathrm{d}I_z(r)}{\mathrm{d}r} = 2\pi r\, j_z(r). \tag{5.4}$$

AMPÈRE's law in integral form relates $B_\varphi(r)$ to the total enclosed current

$$B_\varphi = \frac{\mu_0}{2\pi r} I_z(r) = \frac{\mu_0}{r} \int_0^r j_z(r') r'\,\mathrm{d}r'. \tag{5.5}$$

If the conducting fluid lies almost entirely inside $r = R$, the magnetic flux density outside the plasma is

$$B_\varphi(r) = \frac{\mu_0 I_0}{2\pi r} \quad \text{with} \quad I_0 = \int_0^R j_z(r)2\pi r\,\mathrm{d}r \tag{5.6}$$

where I_0 is the total current flowing inside the cylindrical plasma column. Using the expressions for $B_\varphi(r)$ (5.6) and $j_z(r)$ (5.4) the magnetic pressure has to obey

$$\begin{aligned} \frac{\mathrm{d}p_{\mathrm{mag}}}{\mathrm{d}r} &= -\frac{\mu_0}{4\pi^2 r^2} I_z(r) \frac{\mathrm{d}I_z(r)}{\mathrm{d}r} \\ 4\pi^2 r^2 \frac{\mathrm{d}p_{\mathrm{mag}}}{\mathrm{d}r} &= -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2}\mu_0 I_z^2(r)\right). \end{aligned} \tag{5.7}$$

If we now integrate this equation from 0 to $r = R$ using integration by parts, we obtain

$$4\pi^2 r^2 p(r)\Big|_0^R - 4\pi \int_0^R 2\pi r\, p(r)\,\mathrm{d}r = -\frac{1}{2}\mu_0 I_0^2, \tag{5.8}$$

where I_0 is the total current flowing through the entire cross section. Considering that the plasma pressure is zero for $r \geq R$ and finite for $0 \leq r < R$, the first term of (5.8) vanishes. Therefore we find that

$$\begin{aligned} I_0^2 &= \frac{8\pi}{\mu_0} \int_0^R 2\pi r\, p(r)\,\mathrm{d}r \\ &= \frac{8\pi}{\mu_0} k_B (T_e + T_i) \int_0^R 2\pi n(r)\,\mathrm{d}r, \end{aligned} \tag{5.9}$$

if the partial pressures of electrons and ions are governed by (5.2) with

$$p(r) = p_e(r) + p_i(r) = n(r) k_B (T_e + T_i). \tag{5.10}$$

We can rewrite the result of I_0^2 as

$$I_0^2 = \frac{8\pi}{\mu_0} k_B (T_e + T_i) N_l \quad \text{BENNET-relation} \quad , \tag{5.11}$$

where we introduced the number of particles per unit length of the plasma column as N_l

$$N_l = \int_0^R 2\pi n(r)\,\mathrm{d}r\,. \tag{5.12}$$

The BENNET-relation gives the total current that must be discharged through the plasma column in order to confine a plasma at a specific temperature and given number of particles per length N_l. Usually the current required for the confinement of hot plasmas is very large. As an example suppose that $N_l = 10^{19}\,\frac{1}{\mathrm{m}}$ and a plasma temperature $(T_e+T_i) = 10^8\,\mathrm{K}$. It follows that the required current I_0 is of the order of $10^6\,\mathrm{A}$.

In order to obtain the radial distribution of $p(r)$ in terms of $B_\varphi(r)$, it is convenient to start from (5.3) and proceed differently. From AMPÈRE's $\vec{\boldsymbol{\nabla}} \times \boldsymbol{B} = \mu \boldsymbol{j}$ we have in cylindrical coordinates

$$\begin{aligned} \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\big(rB_\varphi(r)\big) &= \mu_0 j_z(r) \\ \frac{1}{\mu_0}\frac{\mathrm{d}B_\varphi(r)}{\mathrm{d}r} + \frac{1}{\mu_0}\frac{B_\varphi(r)}{r} &= j_z(r). \end{aligned} \tag{5.13}$$

If we substitute this result for $j_z(r)$ into (5.3) this yields

$$\begin{aligned} \frac{\mathrm{d}p(r)}{\mathrm{d}r} &= -\frac{1}{2\mu_0 r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2 B_\varphi^2(r)\Big) \\ p(r) &= p(0) - \frac{1}{2\mu_0}\int_0^r \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2 B_\varphi^2(r)\Big)\mathrm{d}r\,. \end{aligned} \tag{5.14}$$

In particular, since for $r = R$ we have $p(R) = 0$ we can determine $p(0)$ and substitute into (5.14)

$$p(r) = \frac{1}{2\mu_0}\int_r^R \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2 B_\varphi^2(r)\Big)\mathrm{d}r\,. \tag{5.15}$$

The *average* pressure $\bar{p}$ inside the cylinder can be related to the total current I_0 without knowing the detailed radial dependence. It is defined by

$$\bar{p} := \frac{1}{\pi R^2}\int_0^R 2\pi r\, p(r)\,\mathrm{d}r = -\frac{1}{R^2}\int_0^R r^2\frac{\mathrm{d}p(r)}{\mathrm{d}r}\,\mathrm{d}r\,, \tag{5.16}$$

where in the second step we used integration by parts (the integrated term is zero because $p(R) = 0$). Replacing $\frac{\mathrm{d}p}{\mathrm{d}r}$ using equation (5.14) we get

$$\bar{p} = \frac{B_\varphi^2(R)}{2\mu_0} = \frac{\mu_0 I_0^2}{8\pi^2 R^2}. \tag{5.17}$$

This result shows that the average kinetic pressure in the equilibrium is balanced by the magnetic pressure at the boundary.

Examples

First we want to consider the case in which the current density is constant for $r < R$. Taking $j_z = \frac{I_0}{\pi R^2}$ we obtain with (5.6)

$$B_\varphi(r) = \frac{\mu_0 I_0}{\pi R^2 r} \int_0^r \mathrm{d}r = \frac{\mu_0 I_0}{2\pi R^2} r \qquad (r < R). \tag{5.18}$$

Substituting this result into (5.15) we obtain a parabolic pressure dependence versus radius

$$p(r) = \frac{1}{2\mu_0} \int_r^R \frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{\mu_0^2 I_0^2 r^2}{4\pi^2 R^4} \right) \mathrm{d}r = \frac{\mu_0 I_0^2}{4\pi^2 R^2} \left(1 - \frac{r^2}{R^2} \right). \tag{5.19}$$

Note that the axial pressure $p(0)$ is twice the average pressure $\bar{p}$ given in (5.17). The radial dependence of the various quantities is shown in figure 12 (left).

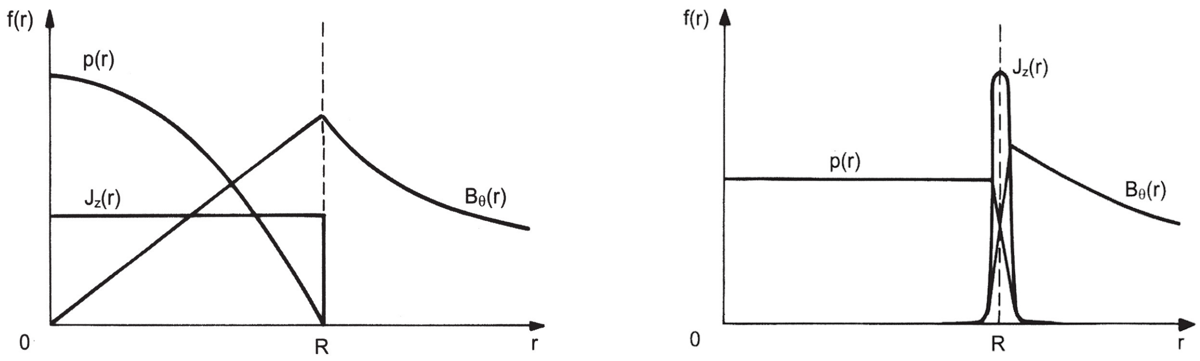


Fig. 12: Left: Radial dependence of $B_\varphi(r)$ and plasma pressure for constant current density $j_z(r)$.
Right: Radial dependence of $B_\varphi(r)$ and plasma pressure with a surface current density $j_z(r)$.

Another radial distribution of $j_z(r)$ is also of interest in the investigation of the equilibrium pinch, in which the current density is confined to a very thin layer on the surface of the column. This model is appropriate for a highly conduction fluid. This surface current density can (in the case of perfect conduction) be conveniently represented by a DIRAC delta function. In this case there is no magnetic field inside the plasma and we only have nonzero $B_\varphi(r)$ for $r > R$. The magnetic flux density is given by

$$B_\varphi = \frac{\mu_0 I_0}{2\pi r} \qquad (r > 0), \tag{5.20}$$

where I_0 is the total axial current. Therefore from (5.14) we have

$$p(r) = p(0) \qquad (0 < r < R) \tag{5.21}$$

so that the plasma kinetic pressure is constant inside the cylindrical column and equal to the average value given in (5.17). The radial dependence is sketched in figure 12 (right). Thus, for a perfectly conduction plasma, the magnetic flux density vanishes inside the column and falls off as $\frac{1}{r}$ outside the column. The plasma kinetic pressure is constant inside and zero outside. The pinch effect can be thought of as due to an abrupt build-up of the magnetic pressure $\frac{B_\varphi^2}{2\mu_0}$ in the region external to the plasma column.